SPECTRAL THEORY AND ITS APPLICATION TO DIFFERENTIAL EIGENVALUE PROBLEMS

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1. Introduction. Eigenvalue problems such as those associated with the differential equation $y''+(\lambda-q(x))y=0$, in which $q(x)$ is real-valued and continuous on $a<x<b$, may be treated as concrete applications of the abstract theory of self-adjoint operators on Hilbert space. The discussion which follows will be in a sense an abridged account of this theory with an indication of how it may be applied to the differential problems mentioned. The treatment will be descriptive and suggestive rather than rigorous. In this form it is hoped the account may be of interest particularly to those who are approaching the study of this subject for the first time.

2. Abstract Hilbert space. An abstract Hilbert space $\mathfrak{H}$ is a set of undefined elements f, g, *etc.*, which conform to certain axioms. These axioms are chosen in such a way as to give to $\mathfrak{H}$ a formal structure similar to that of the vectors of ordinary 3-dimensional space. The axioms and their relationship to ordinary space will be discussed in this section.

The set $\mathfrak{E}_3$ of points of 3-space may be identified with a system $\mathfrak{V}_3$ of vectors, for by choosing at random a point 0 to serve as origin (and null vector), each point f may be identified with a vector extending from 0 to f. Adopting the usual rules of vector algebra for vector addition $(+)$ and multiplication by a scalar $(\cdot)$, $f+g$ and $a\cdot f$ are seen to be elements of $\mathfrak{V}_3$ whenever f and g are elements of $\mathfrak{V}_3$ and a is a real number. With respect to these operations $\mathfrak{V}_3$ is a *linear manifold* in the sense that if f and g are in $\mathfrak{V}_3$, then $af+bg\in\mathfrak{V}_3$, where a and b are any real numbers. With this fact in mind, and in view of our desire to use, in general, complex-valued scalars rather than real-valued ones, we adopt as the first axiom of Hilbert space the following:

AXIOM 1. *Operations* $(+)$ *and* $(\cdot)$ *exist with respect to which* $\mathfrak{H}$ *is a linear manifold, the scalar field being in general the set of complex numbers. (If only real-valued scalars are used the Hilbert space is said to be real.)*

$\mathfrak{V}_3$ possesses in addition a so-called *inner product* operation. If f and g are any two points (vectors), $(f, g)=\|f\|\cdot\|g\|\cos\theta$ defines their inner product. Here θ is the angle between the vectors, and $\|f\|$ and $\|g\|$ denote their positive lengths or *norms*. When $\theta=90°$, $(f, g)=0$ and the vectors are *orthogonal* (*i.e.* perpendicular). When $g=f$, $\|f\|=\sqrt{(f, f)}$ which indicates that the norm can be defined in terms of the inner product. If a rectangular coordinate system is used in $\mathfrak{V}_3$ the inner product may be expressed by $(f, g)=\sum_1^3 a_k b_k$, where (a_1, a_2, a_3) and (b_1, b_2, b_3) are the coordinates of f and g respectively. Clearly, $(f, f)=\|f\|^2=\sum_1^3 a_k^2$. When complex-valued scalars rather than real-valued ones are used a_k^2 must be replaced by $|a_k|^2$ or $a_k\bar{a}_k$, and in keeping with this change we must now write $(f, g)=\sum_1^3 a_k\bar{b}_k$. Thus in the complex case the inner product (f, g)

exhibits the following properties:

(f, g) is complex valued, and is uniquely determined for each pair of elements f, g;

$(f, g) = \overline{(g, f)}$;

$(af, g) = a(f, g)$ whereas $(f, ag) = \bar{a}(f, g)$;

$(f_1+f_2, g) = (f_1, g)+(f_2, g)$ and similarly $(f, g_1+g_2) = (f, g_1)+(f, g_2)$;

$(f, f) \geqq 0$, the equality sign holding only if $f=0$.

With these results in mind we now enunciate:

AXIOM 2. *$\mathfrak{H}$ is an inner product space, the inner product exhibiting the properties listed above. A norm is defined in $\mathfrak{H}$ by the formula* $\|f\| = \sqrt{(f, f)}$, $f \in \mathfrak{H}$.

Once the norm has been defined it is possible to introduce notions of convergence and limit (that is, a topology) in $\mathfrak{H}$. Since the norm $\|f-g\|$ is the analog of the 3-dimensional notion of "distance" between two elements f and g, a sequence $\{f_k\}$ of elements of $\mathfrak{H}$ is said to be convergent (in the Cauchy sense) if and only if $\|f_m-f_n\| \to 0$ as $m, n \to \infty$; f is a limit of such a sequence if $\|f-f_n\| \to 0$ as $n \to \infty$. In $\mathfrak{V}_3$ it is known that all Cauchy convergent sequences have limits in the space. This fact is expressed by saying that $\mathfrak{V}_3$ is complete, and it implies that $\mathfrak{V}_3$ contains all its limit points and is therefore a *closed* linear manifold. This property we now postulate for Hilbert space.

AXIOM 3. *$\mathfrak{H}$ is complete; it is therefore a closed linear manifold.*

An *orthonormal system* in $\mathfrak{V}_3$ is a set of mutually perpendicular vectors of unit length. Such a set can have at most 3 members, and if it has 3 members, say ϕ_1, ϕ_2, ϕ_3, it *spans* the entire space in the sense that every element $f \in \mathfrak{V}_3$ may be expressed in the form $f = \sum_1^3 a_k \phi_k$. The orthonormal property can be easily expressed in the inner product notation, $(\phi_k, \phi_m) = \delta_{km}$, where δ_{km} is 0 when $k \neq m$ and has the value 1 when $k = m$.

AXIOM 4. *$\mathfrak{H}$ is such that it can be spanned by an orthonormal set (i.e. one having the orthonormal property described above) of its own elements; if $\{\phi_k\}$ is such a set, every $f \in \mathfrak{H}$ may be expanded in the form $f = \sum a_k \phi_k$. If the set $\{\phi_k\}$ is infinite, $\mathfrak{H}$ is said to be infinite dimensional, and the series $\sum a_k \phi_k$ converges to f in the manner $\|f - \sum_1^n a_k \phi_k\| \to 0$ as $n \to \infty$. If the set $\{\phi_k\}$ is finite with n members, $\mathfrak{H}$ is finite dimensional (more precisely n dimensional).*

Axioms 1 to 4 define an abstract Hilbert space. Any particular system of elements with associated operations which satisfies these axioms may be regarded as a realization of Hilbert space. Thus $\mathfrak{V}_3$ is an example of a real Hilbert space of 3 dimensions.

The most notable example of an infinite dimensional Hilbert space is the *function class* $L_2[a, b]$. This class consists of all complex-valued functions $f(x)$ on the real interval $[a, b]$ for which $\int_a^b |f(x)|^2 dx < \infty$, the integration being taken in the sense of Lebesgue. In treating this as a Hilbert space the operations $(+)$

and $(\cdot)$ are given their ordinary point-wise meaning, $(f+g)(x)=f(x)+g(x)$, $(a\cdot f)(x)=a\cdot f(x)$, and the function which vanishes identically on $[a, b]$ is taken as the null element. f is considered identical with g in case $\int_a^b|f(x)-g(x)|\,dx=0$. The inner product is defined by $(f, g)=\int_a^b f(x)\overline{g(x)}dx$, and the norm of an element $f(x)$ is therefore given by $\|f\|^2=\int_a^b|f(x)|^2dx$. With these definitions $L_2[a, b]$ may be shown to satisfy axioms 1 to 4. It should be noted that convergence in the Hilbert space $L_2[a, b]$ is in the sense of the mean, for $\|f-f_n\|^2=\int_a^b|f(x)-f_n(x)|^2dx$.

Two important inequalities may be shown to hold in abstract Hilbert space. These are Schwarz's inequality $|(f, g)|\leqq\|f\|\cdot\|g\|$, which in $\mathfrak{V}_3$ results from the fact that $|\cos\theta|\leqq 1$, and the triangle inequality $\|f+g\|\leqq\|f\|+\|g\|$, which in $\mathfrak{V}_3$ expresses an obvious geometric relation in any triangle.

3. Linear manifolds and subspaces. Any subset of a Hilbert space $\mathfrak{H}$ which is closed with respect to the operations $(+)$ and $(\cdot)$ is a linear manifold. If it is closed also with respect to the limiting process it is complete, and is therefore a closed linear manifold. Such closed manifolds are themselves Hilbert spaces, and so are called subspaces of $\mathfrak{H}$.

Any finite dimensional linear manifold in $\mathfrak{H}$ is, like $\mathfrak{V}_3$ itself, a closed manifold (a subspace). An infinite dimensional manifold, however, may or may not be closed. If not, its closure can be taken and thus a subspace obtained.

It should be noted that all linear manifolds in $\mathfrak{H}$ contain the null element 0. As an intuitive aid in studying linear manifolds it is helpful to recall that a straight line through the origin or a plane through the origin are examples of these sets in $\mathfrak{V}_3$. (So also are $\mathfrak{V}_3$ itself and the set consisting of the origin alone.)

4. Self-adjoint operators. In a Hilbert space $\mathfrak{H}$ a linear operator T defines a mapping of a linear manifold $\mathfrak{D}\subset\mathfrak{H}$ into an "image" set $\mathfrak{R}\subset\mathfrak{H}$. Moreover the mapping $\mathfrak{D}\to\mathfrak{R}$ is single-valued in that each f in $\mathfrak{D}$ has a unique image g in $\mathfrak{R}$, where g is denoted by Tf. Furthermore $T(a_1f_1+a_2f_2)=a_1Tf_1+a_2Tf_2$ for all elements f_1, f_2 in $\mathfrak{D}$ and for all complex numbers a_1, a_2. $\mathfrak{D}$ is called the *domain* of definition of T, and $\mathfrak{R}$, being the set $T(\mathfrak{D})$, is called the *range* (*i.e.* reach) of T. It is easily seen that $\mathfrak{R}$ also is a linear manifold. It should be noted that $T0=0$.

It may happen that two elements f_1, and f_2 have the same image g. If so, the equation $Ty=0$ has, in addition to the trivial solution $y=0$, a non-zero solution $y=f_1-f_2$, for clearly $T(f_1-f_2)=Tf_1-Tf_2=g-g=0$. This being so, every equation $Ty=g$ has at least two distinct solutions, for if $y=f$ is a solution so is $y=f+f_1-f_2$. The mapping $\mathfrak{D}\to\mathfrak{R}$ is then many-one throughout. In all cases then the mapping $\mathfrak{D}\to\mathfrak{R}$ is either one-to-one or many-one throughout. If it is one-to-one the reverse mapping $\mathfrak{R}\to\mathfrak{D}$ defines the *inverse operator* T^{-1} which is also linear. The non-existence of T^{-1} implies that the equation $Ty=0$ has a non-trivial solution; its existence implies that $Ty=0$ has no solution other than $y=0$ and that $Ty=g$ has a unique solution $y=T^{-1}g$.

A linear operator T is said to be *closed* if $T\lim f_n=\lim Tf_n$ whenever $\{f_n\}$

and $\{Tf_n\}$ are both convergent sequences in $\mathfrak{D}$. A linear operator T is said to be *bounded* if a constant $M>0$ exists such that $\|Tf\| \leq M\|f\|$ for all elements f in $\mathfrak{D}$.

It may be shown that for every linear operator T whose domain is dense in $\mathfrak{H}$ (*i.e.* the closure of $\mathfrak{D}$ is $\mathfrak{H}$) there exists a uniquely determined linear operator T^*, called the *adjoint* of T, such that $(Tf_1, f_2)=(f_1, T^*f_2)$ for every f_1 in the domain of T and every f_2 in the domain of T^*. Moreover T^* is closed. When $T=T^*$, T is said to be *self-adjoint*. A self-adjoint operator T is thus a linear operator such that (i) $\mathfrak{D}$ is dense in $\mathfrak{H}$, (ii) T is closed, (iii) $(Tf_1, f_2)=(f_1, Tf_2)$ for all elements f_1, f_2 in $\mathfrak{H}$. The third condition imparts to T a symmetric character.

5. The spectrum of a self-adjoint operator. From this point on the operator T will be assumed to be self-adjoint. With a given T we can associate a family of operators T_λ, where $T_\lambda f=Tf-\lambda f$ for each complex λ. T_λ is linear (but not necessarily self-adjoint) and its domain of definition is obviously the same as that of T, *i.e.* $\mathfrak{D}$. Its range $\mathfrak{R}_\lambda$ will depend on the choice of λ.

The points λ of the complex λ-plane may now be classified according to the behavior of T_λ as follows:

I. A point λ at which T^{-1} fails to exist is said to belong to the *point spectrum* of the operator T.

II. A point λ at which T^{-1} exists and is an unbounded operator is said to belong to the *continuous spectrum* of T.

III. The points of the λ-plane that are not in the sets I or II are said to belong to the *resolvent set* of T. If λ is in this set, T_λ^{-1} exists and is a bounded operator. The points of the sets I and II together constitute the *spectrum* of T.

The points of the point spectrum are called eigenvalues (characteristic values). If λ_1 is an eigenvalue $T_{\lambda_1}^{-1}$ does not exist and hence the equation $T_{\lambda_1}y=0$ (*i.e.* $Ty-\lambda_1 y=0$) has a non-zero solution f_1. Such a solution is called an eigensolution (characteristic solution). The symmetric relation $(Tf_1, f_1)=(f_1, Tf_1)$ may then be written $\lambda_1(f_1, f_1)=\bar{\lambda}_1(f_1, f_1)$. But $(f_1, f_1)=\|f_1\|^2\neq 0$ and hence $\lambda_1=\bar{\lambda}_1$, so that λ_1 is real. This proves that the points of the point spectrum lie on the real axis of the complex λ-plane.

If λ is not on the real axis it cannot be an eigenvalue and hence T_λ^{-1} exists. The equation $T_\lambda y=g$, $g\in\mathfrak{R}$ must therefore have a unique solution $y=T_\lambda^{-1}g$. We shall show that T_λ^{-1} is a bounded operator, and thus infer that the non-real points of the λ-plane belong to the resolvent set. Since $(T_\lambda y, y)=(Ty-\lambda y, y)=(Ty, y)-\lambda(y, y)$ and $(y, T_\lambda y)=(y, Ty)-\bar{\lambda}(y, y)$, and since (Ty, y) and (y, Ty) are identical we find that $(T_\lambda y, y)-(y, T_\lambda y)=(\bar{\lambda}-\lambda)(y, y)=(\bar{\lambda}-\lambda)\|y\|^2$. On applying the triangle inequality and Schwarz's inequality to this we obtain $|\bar{\lambda}-\lambda|\,\|y\|^2\leq 2\|T_\lambda y\|\cdot\|y\|$, from which it follows that $\|y\|\leq M\|Ty\|$ where $M=2/|\bar{\lambda}-\lambda|$. On substituting $y=T_\lambda^{-1}g$ and $T_\lambda y=g$ in this inequality we obtain finally $\|T^{-1}g\|\leq M\|g\|$ which proves that T_λ^{-1} is bounded. Since the non-real values of λ are in the resolvent set we conclude that the continuous spectrum as well as the point spectrum must lie on the real axis of the λ-plane.

We shall show next that the eigenvalues are at most denumerable. Corresponding to a given eigenvalue λ_k there will exist a linear manifold of eigensolutions satisfying $T_{\lambda_k} y = 0$, for clearly any finite linear combination of solutions of this homogeneous equation is also a solution. ($y = 0$ is included in this manifold although it is not strictly an eigensolution). The closed manifold of these solutions will be denoted by $\mathfrak{M}_k$. Now consider two distinct eigenvalues λ_k and λ_m with manifolds $\mathfrak{M}_k$ and $\mathfrak{M}_m$ respectively. If $f_k \in M_k$ and $f_m \in M_m$ then $Tf_k = \lambda_k f_k$ and $Tf_m = \lambda_m f_m$ and hence the symmetric relation $(Tf_k, f_m) = (f_k, Tf_m)$ may be written $\lambda_k(f_k, f_m) = \bar{\lambda}_m(f_k, f_m)$. But $\bar{\lambda}_m = \lambda_m \neq \lambda_k$ and hence $(f_k, f_m) = 0$. This proves that the eigensolution manifolds form a mutually orthogonal system. The manifolds can therefore be at most denumerable in number (in view of the fact that $\mathfrak{H}$ has a countable number of dimensions), and hence the eigenvalues themselves are at most denumerable. These points therefore may be arranged as a sequence $\{\lambda_k\}$. As one might expect from its name the continuous spectrum is usually an interval or set of intervals although in some cases it may degenerate into a single point.

6. Spectral representation of T, and related formulae. Let $\mathfrak{M}$ denote the closed linear manifold spanned by the entire set of eigensolution manifolds $\mathfrak{M}_k$. Every element f in $\mathfrak{M}$ can then be uniquely decomposed into components in the subspaces $\mathfrak{M}_k$ (by orthogonal projection). Thus if $f \in \mathfrak{M}$ we can write $f = \sum f_k$, $f_k \in \mathfrak{M}_k$. Here, and later where similar expansions occur, it is to be understood that the summation is taken over the entire set of subspaces $\mathfrak{M}_k$.

Let $\mathfrak{N}$ be the orthogonal complement of $\mathfrak{M}$. By this we mean that $\mathfrak{N}$ is the closed linear manifold of elements in $\mathfrak{H}$ which are orthogonal to $\mathfrak{M}$. (In $\mathfrak{V}_3$ we could think of the x and y-axes as being the manifolds $\mathfrak{M}_k$, the xy-plane as being $\mathfrak{M}$, and the z-axis as being $\mathfrak{N}$.) If ψ is in $\mathfrak{H}$ it can be uniquely decomposed in the form $\psi = \psi_1 + \psi_2$ where $\psi_1 \in \mathfrak{M}$ and $\psi_2 \in \mathfrak{N}$.

Let T_1 denote the operator T when applied solely in $\mathfrak{M}$, and let T_2 denote it when applied solely in $\mathfrak{N}$. Then $T\psi = T_1\psi_1 + T_2\psi_2$ if $\psi \in \mathfrak{D}$. The domains $\mathfrak{D}_1$, $\mathfrak{D}_2$ of T_1, T_2 are then the sets of all elements ψ_1, ψ_2 (respectively) obtained when $\psi = \psi_1 + \psi_2$ runs through the set $\mathfrak{D}$.

Let us consider first the operator T_1. If $y \in \mathfrak{D}_1$, $y = \sum y_k$ and $T_1 y = T_1 \sum y_k = \sum T_1 y_k = \sum \lambda_k y_k$. This shows, incidentally, that the range of T_1 is also in $\mathfrak{M}$. The interchange of $\sum$ and T_1 implies that $\sum T_1 y_k = \sum \lambda_k y_k$ is convergent, and this in turn implies that $\sum |\lambda_k|^2 \|y_k\|^2 < \infty$. This latter condition is in fact the defining condition for the domain $\mathfrak{D}_1$. The formula proved above,

$$T_1 y = \sum \lambda_k y_k, \tag{1}$$

gives the spectral representation of T_1.

Another useful formula may be deduced from (1). Suppose λ is not in the point spectrum of T_1 (*i.e.* $\lambda \neq \lambda_k$ for all k). Then $T_{1\lambda}^{-1}$ exists ($T_{1\lambda} y$ denotes $T_1 y - \lambda y$), and the equation $T_{1\lambda} y = p$, where p is a given element in the range of $T_{1\lambda}$, has a unique solution $y = T_{1\lambda}^{-1} p$. But $y = \sum y_k$, and a similar expansion $p = \sum p_k$ holds

for p. Hence $T_{1\lambda}y=p$ (*i.e.* $T_1y-\lambda y=p$) may be written $\sum\lambda_k y_k-\lambda\sum y_k=\sum p_k$. Since the decomposition into $\mathfrak{M}_k$ components is unique these series can be equated term by term to give $(\lambda_k-\lambda)y_k=p_k$. On solving this for y_k and summing the result we obtain

$$y = T_{1\lambda}^{-1}p = -\sum\frac{p_k}{\lambda-\lambda_k}. \tag{2}$$

Formula (2) indicates that $y=y(\lambda)$ regarded as a function of the complex variable λ is analytic throughout the λ-plane except for poles at the eigenvalues $\lambda=\lambda_k$. Moreover, $-p_k$ is the residue of the pole at $\lambda=\lambda_k$ and hence the expansion $p=\sum p_k$ may be expressed as a summation involving the negatives of the residues of the poles λ_k. In the applications it frequently happens that $y(\lambda)$ can be computed by direct means; the eigenvalues of T_1 and the related eigensolution expansion of p can then be obtained from this result. From (2) it may be deduced also that $T_{1\lambda}^{-1}$ is bounded when $\lambda\neq\lambda_k$, and hence it may be inferred that the continuous spectrum of T_1 is empty. Thus we conclude that T_1 is a self-adjoint operator in the subspace $\mathfrak{M}$ and its spectrum is a pure point spectrum.

When it comes to the consideration of the operator T_2 in the Hilbert space $\mathfrak{N}$ we find the conditions quite changed. There are no eigenvalues now since these were all accounted for in the subspace $\mathfrak{M}$, and the arguments used above therefore have little meaning. Formulae (1) and (2) if re-stated in Stieltjes integral form and re-interpreted may, however, be used to describe T_2. That this is so is the assertion of one of the fundamental theorems in the theory of self-adjoint operators. For an outline of a proof of this theorem the reader is referred to [4]. We shall be content here merely to describe the result.

Let $E_1(t)$ be a family of linear operators defined on $-\infty<t<\infty$ with domain $\mathfrak{M}$ by the relation $E_1(t)y=\sum_{\lambda_j\leq t}y_j$, $y\in\mathfrak{M}$. Here the notation means that $E_1(t)y$ is the partial sum of the components y_j corresponding to the eigenvalues λ_j which lie to the left or are coincident with t. $E_1(t)y$ has the nature of a step-function, with steps y_k occurring at the points $t=\lambda_k$. From the theory of Stieltjes integrals it follows that $\int_{-\infty}^{\infty}dE_1(t)y=\sum y_k=y$, and for this reason $E_1(t)$ is called a *resolution of the identity* in the subspace $\mathfrak{M}$. Formulae (1) and (2) may now be written

$$T_1y=\int_{-\infty}^{\infty}t\,dE_1(t)y, \qquad y\in\mathfrak{D}_1, \tag{1}'$$

$$y=T_{1\lambda}^{-1}p=-\int_{-\infty}^{\infty}\frac{dE_1(t)p}{\lambda-t}, \qquad p\in\mathfrak{R}_{1\lambda}. \tag{2}'$$

In dealing with T_2 of the subspace $\mathfrak{N}$ we shall need a new family of operators $E_2(t)$ with domain $\mathfrak{N}$. The operators $E_2(t)$ must be such that $\int_{-\infty}^{\infty}dE_2(t)y=y$

for all elements y in $\mathfrak{N}$. Also, the real numbers t which are not in the spectrum of T_2 must be points of constancy of $E_2(t)$ (a point of constancy being defined as one in a neighborhood of which $E_2(t)$ is constant), so that these points contribute nothing to the integral representation of y. On the other hand, every neighborhood Δ of a point of the (continuous) spectrum of T_2 must be one in which $E_2(t)$ varies. The vector $\int_\Delta dE_2(t)y$ may be interpreted as the projection of y on a closed linear manifold $\mathfrak{N}_\Delta$ associated with Δ and it may be shown that any two such manifolds corresponding to distinct intervals Δ_1, Δ_2 are orthogonal. There is thus a strong analogy between this case and that of the operator T_1, in which, for a neighborhood Δ of a single eigenvalue λ_k, $\int_\Delta dE_1(t)y = y_k \in \mathfrak{M}_k$, and manifolds $\mathfrak{M}_k$, $\mathfrak{M}_m$ corresponding to distinct eigenvalues λ_k, λ_m are orthogonal. Formulae (1)′ and (2)′ may now be re-stated for the subspace $\mathfrak{N}$ by changing the subscripts from 1 to 2 and replacing $p \in \mathfrak{R}_{1\lambda}$ by $q \in \mathfrak{R}_{2\lambda}$.

The results concerning T_1 and T_2 may now be combined to apply to the operator T in $\mathfrak{H}$. We define $E(t)$ with domain $\mathfrak{H}$ by the statement $E(t)\psi = E_1(t)\psi_1 + E_2(t)\psi_2$, $\psi \in \mathfrak{H}$, where ψ_1 and ψ_2 are the components of ψ in the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ respectively. We then have $\int_{-\infty}^{\infty} dE(t)\psi = \int_{-\infty}^{\infty} dE_1(t)\psi_1 + \int_{-\infty}^{\infty} dE_2(t)\psi_2 = \psi_1 + \psi_2 = \psi$, and thus we see that $E(t)$ is a resolution of the identity in $\mathfrak{H}$. We can now write*

$$Ty = \int_{-\infty}^{\infty} t dE(t)\, y, \qquad y \in \mathfrak{D}. \tag{3}$$

$$y = y(\lambda) = T_\lambda^{-1} g = -\int_{-\infty}^{\infty} \frac{dE(t)g}{\lambda - t}, \qquad g \in \mathfrak{R}_\lambda. \tag{4}$$

The domain $\mathfrak{D}$ consists of all elements y for which $\int_{-\infty}^{\infty} t^2 d\|E(t)y\|^2 < \infty$. (This is in keeping with the observation made earlier in the case of T_1 that $\sum T_1 y_k = \sum \lambda_k y_k$ must converge and therefore $\sum |\lambda_k|^2 \|y_k\|^2 < \infty$.) The range $\mathfrak{R}_\lambda$ is a proper subset of $\mathfrak{H}$ when λ is in the continuous spectrum but is identical with $\mathfrak{H}$ when λ is in the resolvent set. In connection with (4) it should be noted that the singularities of $y(\lambda)$ determine the spectrum of T, the poles being the eigenvalues and other singularities determining the continuous spectrum.

We have considered the eigensolution expansion $p = \sum p_k$ in the subspace $\mathfrak{M}$ and have discovered that it may be expressed in terms of the residues of the poles of $y(\lambda)$ the unique solution of $T_{1\lambda} y = p$. We now ask, what is the nature of the "expansion" of an element q in the subspace $\mathfrak{N}$? The answer is—it is a continuous summation (*i.e.* an integral) rather than a series. To obtain a formula for this expansion let us look again at the analogous problem in the subspace $\mathfrak{M}$.

* The integral operators such as T, T_λ^{-1} are not Stieltjes integrals in the ordinary sense of course, but may be defined in terms of such integrals. Thus T may be defined as the (unique) linear operator for which $(Tf, g) = \int_{-\infty}^{\infty} t\, d(E(t)f, g)$ for all pairs of elements $f \in \mathfrak{D}$, $g \in \mathfrak{H}$. A similar definition may be given for T_λ^{-1}.

If $y(\lambda)$ of formula (2) is integrated around a contour Γ_α in the λ-plane enclosing the poles λ_k lying between $-\alpha$ and α and excluding all the others, the result is $\int_{\Gamma_\alpha} y(\lambda)d\lambda = 2\pi i \sum_{-\alpha<\lambda_j<\alpha} (-p_j)$. From this we deduce the formula

$$p = \sum p_k = \lim_{\alpha\to\infty} -\frac{1}{2\pi i}\int_{\Gamma_\alpha} y(\lambda)d\lambda.$$

Γ_α may be taken to be a rectangle with vertical sides through the points $-\alpha$ and α and horizontal sides through the points $-\beta i$ and βi, $\beta>0$. The vertical pieces of the contour may be neglected if we let $\beta\to 0+$. On the upper line we will have $\lambda=\mu+\beta i$ and $d\lambda=d\mu$ where the real variable μ changes from α to $-\alpha$; on the lower line we will have $\lambda=\mu-i\beta$, $d\lambda=d\mu$, where μ varies from $-\alpha$ to α. It follows then that

$$p = \lim_{\alpha\to\infty} \lim_{\beta\to 0} \frac{1}{2\pi i}\int_{\Gamma_\alpha} [y(\mu+i\beta) - y(\mu-i\beta)]d\mu. \tag{5}$$

This formula carries over without essential change to the case of T_2 in the subspace $\mathfrak{N}$, and therefore to the case of T in the space $\mathfrak{H}$. In the first instance, for the expansion of an element q, $y(\lambda)$ must be identified with the unique solution of $T_{2\lambda}y=q$. In the general case, for the expansion of an element g, $y(\lambda)$ must be identified with the unique solution of $T_\lambda y=g$. The expansion of g in $\mathfrak{H}$ will in fact be the sum of the expansions of its components, p in $\mathfrak{M}$ and q in $\mathfrak{N}$.

7. Application to differential problems. Consider the differential equation $y''+(\lambda-q(x))y=0$, in which $q(x)$ is real-valued and continuous in $a<x<b$, its behavior at a and b being arbitrary. Let the operator T be defined by the relation $Ty=-y''+q(x)y$, its domain of definition $\mathfrak{D}$ being chosen so that y and Ty both are in the Hilbert space $L_2[a, b]$ when $y\in\mathfrak{D}$. Let $\mathfrak{D}$ be restricted further if necessary so as to insure that T is self-adjoint.

What this restriction involves may be seen by imposing the symmetric condition $(Ty_1, y_2)=(y_1, Ty_2)$, in which $y_1(x)$, $y_2(x)$ are any two functions in $L_2[a, b]$. We have

$$(Ty_1, y_2) - (y_1, Ty_2) = \int_a^b (-y_1'' + qy_1)\bar{y}_2 dx - \int_a^b y_1(\overline{-y_2'' + qy_2})dx$$

$$= \int_a^b (y_1\bar{y}_2'' - y_1''\bar{y}_2)dx = W_b(y_1\bar{y}_2) - W_a(y_1\bar{y}_2),$$

where $W_x(y_1\bar{y}_2)=y_1\bar{y}_2'-y_1'\bar{y}_2$ is the Wronksian of y_1 and $\bar{y}_2$ evaluated at the point x. To insure the vanishing of this result we impose on the functions $y(x)\in\mathfrak{D}$ a pair of linear homogeneous boundary conditions $\gamma y(a)+\gamma' y'(a)=0$, $\delta y(b)+\delta' y'(b)=0$, in which γ, γ', δ, δ' are preassigned real constants. The first condition insures that $W_a(y_1\bar{y}_2)=0$, and the second that $W_b(y_1\bar{y}_2)=0$. With $\mathfrak{D}$

so restricted it can be verified that $\mathfrak{D}$ is dense in $\mathfrak{H}$ and that T is a closed operator. It follows then that T is self-adjoint.

It should be noted that when $a=-\infty$, or when a is finite and $q(a+0)$ is infinite, the L_2 requirements in $\mathfrak{D}$ are enough to make $W_a(y_1\bar{y}_2)=0$. No boundary condition is needed at the point a in this case. A similar observation can be made about the point b. Thus, for example, in the case when $q(x)$ is continuous in $-\infty<x<\infty$ and $L_2[-\infty, \infty]$ is the Hilbert space no boundary conditions are needed to make T self-adjoint.

8. Three illustrative examples.

Example 1. Suppose $q(x)\equiv 0$ on $0\leqq x\leqq 1$, and let $T\equiv -d^2/dx^2$, and let $\mathfrak{D}$ be restricted by the boundary conditions $y(0)=y(1)=0$. T is then a self-adjoint operator in the Hilbert space $L_2[0, 1]$, and the equation $T_\lambda y=g=-f(x)$ is equivalent to the differential system $y''+\lambda y=f(x)$, $y(0)=y(1)=0$. An easy computation shows that the unique solution of this system is

$$y(\lambda)=y(x,\lambda)=\frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}\sin\sqrt{\lambda}}\int_0^x f(t)\sin\sqrt{\lambda}\,t\,dt + \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}\sin\sqrt{\lambda}}\int_x^1 f(t)\cos\sqrt{\lambda}(1-t)\,dt.$$

The only singularities of $y(\lambda)$ are the poles $\lambda_k=k^2\pi^2$, $k=1, 2, 3, \cdots$. The continuous spectrum is therefore empty and the poles are the eigenvalues. The residue of the pole $\lambda_k=k^2\pi^2$ is easily computed, and is found to be $-2\sin k\pi x\int_0^1 f(t)\sin k\pi t\,dt$. The eigensolution expansion of $f(x)$ is therefore

$$f(x)=2\sum_1^\infty \sin k\pi x\int_0^1 f(t)\sin k\pi t\,dt.$$

This will be recognized as the Fourier sine expansion of $f(x)$ on the half-period interval $(0, 1)$. The convergence of the series is of course in the sense of the mean.

Example 2. Suppose $q(x)\equiv 0$ on $-\infty<x<\infty$, and let $T\equiv -d^2/dx^2$. No boundary conditions are needed; the L_2 requirements are enough to insure that T is a self-adjoint operator in the Hilbert space $L_2[-\infty, \infty]$.

The equation $T_\lambda y=g=-f(x)$ is equivalent to the differential system $y''+\lambda y=f(x)$ $y\in\mathfrak{D}$, and the unique solution of this is given by

$$y(\lambda)=y(x,\lambda)=-\frac{1}{2i\sqrt{\lambda}}\left[\int_{-\infty}^x f(t)e^{i\sqrt{\lambda}(x-t)}dt+\int_x^\infty f(t)e^{-i\sqrt{\lambda}(x-t)}dt\right].$$

This function has no poles and so the point spectrum is empty. The expansion of $g=-f(x)$ may be obtained by an application of (5) in which $y(\lambda)$ is as given above.

$$f(x) = -\lim_{\alpha\to\infty}\lim_{\beta\to 0}\frac{1}{2\pi i}\int_{-\alpha}^{\alpha}[y(x,\mu+i\beta)-y(x,\mu-i\beta)d\mu.]$$

When $\mu<0$ the integrand$\to 0$ as $\beta\to 0$, and hence only the range of integration from 0 to α need be considered. When $\mu>0$ we find, on taking the limits as $\alpha\to\infty$, $\beta\to 0$, and substituting $\mu=\xi^2$, that

$$f(x) = \frac{1}{\pi}\int_0^{\infty}d\mu[\cos\xi x\int_{-\infty}^{\infty}f(t)\cos\xi t dt + \sin\xi x\int_{-\infty}^{\infty}f(t)\sin\xi t dt].$$

This is the familiar Fourier Integral expansion of $f(x)$ on $-\infty<x<\infty$.

Example 3. We consider finally a case of the more general type in which $q(x)\not\equiv 0$. Suppose $q(x)=x^2$ on $-\infty<x<\infty$ and let T be defined by $Ty=-y''+x^2y$, with $\mathfrak{D}$ appropriately chosen (as described above) in the Hilbert space $L_2[-\infty,\infty]$. The equation $T_\lambda y=-f(x)$ is then equivalent to $y''+(\lambda-x^2)y=f(x)$, $y\in\mathfrak{D}$. The unique solution of the latter is expressible in terms of the function

$$\phi(x,\lambda) = e^{-\frac{1}{2}x^2}\int_{\infty}^{(0+)}e^{-x\xi-\frac{1}{2}\xi^2}\xi^{-\frac{1}{2}\lambda-\frac{1}{2}}d\xi,$$

[see [3], p. 61] as follows:

$$y = y(x,\lambda) = \frac{1}{\omega(\lambda)}\int_{-\infty}^{x}f(t)\phi(x,\lambda)\phi(-t,\lambda)dt + \frac{1}{\omega(\lambda)}\int_{x}^{\infty}f(t)\phi(t,\lambda)\phi(-x,\lambda)dt,$$

where $\omega(\lambda)=W_x[\phi(-x,\ \lambda),\ \phi(x,\ \lambda)]=-(1+e^{-\pi i\lambda})^{\frac{1}{2}}2^{\frac{1}{2}-\frac{1}{2}\lambda}\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}\lambda)$. The only singularities of this function are the simple poles $\lambda_k=2k+1$, $k=0$, 1, 2, $\cdots$. These are the eigenvalues. The residue of the pole $\lambda_k=2k+1$ is found to be

$$\gamma_n e^{-\frac{1}{2}x^2}H_n(x)\int_{-\infty}^{\infty}e^{-\frac{1}{2}t^2}H_n(t)f(t)dt,$$

where $\gamma_n^{-1}=2^n n!\pi^{\frac{1}{2}}$, and the functions $H_n(x)$ are the Hermite polynomials. The eigensolution expansion of $f(x)$ is therefore

$$f(x) = \sum_0^{\infty}\gamma_n e^{-\frac{1}{2}x^2}H_n(x)\int_{-\infty}^{\infty}e^{-\frac{1}{2}t^2}H_n(t)f(t)dt.$$

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AN EXTENDED INVERSIVE GEOMETRY

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1. Introduction. The Inversive Geometry of Morley and Morley [8], Weaver, and others is based on the complex number $z=x+iy$ and its conjugate $\bar{z}=x-iy$, where i is defined by the relation $i^2=-1$. Using this notation the above authors have proved theorems concerning the geometry of a triangle inscribed in a circle and the Line of Images of a point on a circle with respect to the sides of an inscribed triangle. By extending the definition of the variable z it is possible to state and prove corresponding theorems relating to triangles inscribed in central conics, and also to show that the Line of Images is a special case of a more general Line of Images of a point on a conic with respect to the sides of an inscribed triangle.

It is the purpose of this paper, first, to extend the concept of perpendicular diameters of circles by showing them to be special instances of conjugate diameters of central conics; second, to define a nine-point conic of a triangle inscribed in a central conic, to obtain its equation, and to exhibit a pseudo Euler Line which passes through four points analogous to the four points of the Euler Line of a triangle, third, to define a general Line of Images, and a general Simson Line.

2. The notation. Let the variables z and $\bar{z}$ be

$$z = x + ry \quad \text{and} \quad \bar{z} = x - ry, \tag{1}$$

where r is defined by one of the relations

$$r^2 = -1, \quad r^2 = -k^2, \quad r^2 = 1, \quad r^2 = k^2 \tag{2}$$

where k is a real number. Then the equation $z\bar{z}=1$ becomes in ordinary rectangular Cartesian coordinates either

$$\begin{aligned} x^2 + y^2 &= 1, \text{ the unit circle,} \\ x^2 + k^2y^2 &= 1, \text{ an ellipse,} \\ x^2 - y^2 &= 1, \text{ a rectangular hyperbola, or} \\ x^2 - k^2y^2 &= 1, \text{ a hyperbola.} \end{aligned} \tag{3}$$

3. Pseudo altitudes of triangle. Let t_1, t_2, t_3 where $t_i = x_i + ry_i$ and $\bar{t}_i = x_i - ry_i$, be the vertices of a triangle inscribed in the central conic $z\bar{z} = 1$. First consider the case where $r^2 = -1$. Now z is the ordinary complex number and the geometry the Inversive Geometry of Morley and Morley. The equation of the side t_1, t_2 of the triangle is [7]

$$z + t_1 t_2 \bar{z} = t_1 + t_2, \tag{4}$$

and the equation of the altitude from t_3 upon this side is [7]

$$z - t_1 t_2 \bar{z} = t_3 - t_1 t_2 \bar{t}_3. \tag{5}$$

The foot of the altitude from t_3 upon this side is obtained by solving these equations for z. Thus adding,

$$2z = t_1 + t_2 + t_3 - t_1 t_2 \bar{t}_3, \tag{6}$$

or

$$z = S_1/2 - t_1 t_2/(2t_3), \tag{7}$$

where $S_1 = t_1 + t_2 + t_3$. In like manner the equations of the altitudes from t_1 and t_2 are, respectively,

$$z - t_2 t_3 \bar{z} = t_1 - t_2 t_3 \bar{t}_1 \tag{8}$$

and

$$z - t_3 t_1 \bar{z} = t_2 - t_3 t_1 \bar{t}_2. \tag{9}$$

The feet of these altitudes are, respectively,

$$S_1/2 - t_2 t_3/(2t_1) \quad \text{and} \quad S_1/2 - t_3 t_1/(2t_2). \tag{10}$$

To find the point of intersection of altitudes (8) and (9), multiply the first equation by t_1, the second by t_2 and subtract. This gives, remembering that $t_1 \bar{t}_1 = t_2 \bar{t}_2 = 1$,

$$z(t_1 - t_2) = t_1^2 - t_2^2 + t_3 t_1 - t_2 t_3. \tag{11}$$

Dividing through by $t_1 - t_2$

$$z = t_1 + t_2 + t_3, \tag{12}$$

or

$$z = S_1. \tag{13}$$

By substitution we find that this value of z satisfies equation (5). Hence the three altitudes of the triangle are concurrent at the point S_1. This point is called the orthocenter of the triangle.

Now let us examine the geometry of equations (4) and (5) for the other definitions of r. The algebraic operations are identical. The equation of the side

t_1, t_2 of a triangle inscribed in the conic corresponding to the particular definition of r is (4) [7]. In order to identify the line represented by (5), consider the equations

$$z + t_1 t_2 \bar{z} = 0 \tag{14}$$

and

$$z - t_1 t_2 \bar{z} = 0. \tag{15}$$

These are the equations of two diameters of the conic which are parallel to lines (4) and (5), respectively. Equation (15) is satisfied by $(t_1+t_2)/2$, the mid-point of the chord t_1, t_2 of the conic. Line (14) is parallel to this chord. Hence by definitions found in Analytic Geometry (14) and (15) are conjugate diameters of the conic. Thus lines (4) and (5) are parallel, respectively, to a pair of conjugate diameters of the conic. These lines are called conjugate lines with respect to the conic.

Any pair of lines L_1 and L_2 which are parallel respectively to a pair of conjugate diameters l_1 and l_2 of the conic will be called conjugate lines. Line L_1 is said to be a conjugate of line L_2 with respect to the conic. Thus the altitude of a triangle inscribed in a circle is a conjugate of the corresponding side of the triangle. Hence if $r^2 = -1$ any two perpendicular lines may be regarded as conjugate lines with respect to the circle.

For values of r other than $r^2 = -1$, line (5) is called a pseudo altitude from the vertex t_3 upon the side t_1, t_2 of the triangle. A pseudo altitude and the corresponding side of the inscribed triangle form a pair of conjugate lines with respect to the circumscribing conic. Evidently every triangle inscribed in a conic has three pseudo altitudes. The feet of the pseudo altitudes upon the corresponding sides of the triangle are

$$\begin{aligned} &S_1/2 - t_2 t_3/(2t_1), && \text{from } t_1, \\ &S_1/2 - t_3 t_1/(2t_2), && \text{from } t_2, \\ &S_1/2 - t_1 t_2/(2t_3), && \text{from } t_3. \end{aligned} \tag{16}$$

The pseudo altitudes are concurrent at a point H' whose coordinate is S_1. H' is called the pseudo orthocenter of the inscribed triangle with respect to the circumscribing conic. A pseudo altitude or any line parallel to it is a conjugate of the corresponding side of a triangle inscribed in a conic.

4. The nine-point conic. The reader will recall that the nine-point circle of a triangle passes through the following sets of points, (a) the mid-points of the sides, (b) the feet of the altitudes, (c) the mid-points of the line segments joining the orthocenter H to the vertices of the triangle. Similarly, there is a nine-point conic of a triangle inscribed in a central conic, which passes through, (a) the mid-points of the sides, (b) the feet of the pseudo altitudes, (c) the mid-points of the line segments joining the pseudo orthocenter H' to the vertices of

the triangle. This statement is easily proved by using the Extended Inversive Geometry described in the Introduction. Let

$$z\bar{z} = 1, \tag{17}$$

be the equation of a central conic and let t_1, t_2, t_3 be the coordinates of the vertices of the triangle. The coordinates of the mid-points of the sides are

$$(t_1 + t_2)/2, \quad (t_2 + t_3)/2 \quad \text{and} \quad (t_3 + t_1)/2. \tag{18}$$

The coordinates of the feet of the pseudo altitudes on the opposite sides are given by (16) in Section 3. Since the coordinate of H' is S_1, the mid-points of the line segments joining H' to each vertex are

$$(S_1 + t_1)/2, \qquad (S_1 + t_2)/2, \qquad (S_1 + t_3)/2. \tag{19}$$

These nine points all lie on the conic whose equation is

$$(z - S_1/2)(\bar{z} - \overline{S}_1/2) = 1/4. \tag{20}$$

This can be verified by substituting the coordinates of each point in the equation. The coordinate of the center N of the nine-point conic is $S_1/2$. Its axes are parallel to and one half the length of the axes of the circumscribing conic. The eccentricities are equal.

5. The pseudo Euler line. The center O of the conic, the centroid G of the triangle, the center N of the nine-point conic and the pseudo orthocenter H' are collinear. This follows from the fact that the coordinates of these points are, respectively, 0, $S_1/3$, $S_1/2$, and S_1. Due to its similarity to the Euler Line of a triangle we are calling this line the pseudo Euler Line.

6. Projective transformation. In the case of the nine-point conic of a triangle inscribed in an ellipse the configuration can also be obtained by the projective transformation $x = \bar{x}$ and $y = k\bar{y}$, applied to the nine-point circle configuration.

7. The line of images. The concept of conjugate lines can be used to unify the theory of the line of images. The image of a point T of a circle with respect to the side t_2, t_3 of an inscribed triangle has been defined as a point T_1 such that the side of the triangle is the perpendicular bisector of the line segment TT_1 [4]. The images of the point T with respect to the three sides of the triangle are collinear. The equation of this line is [5]

$$Tz - S_3\bar{z} = TS_1 - S_2, \tag{21}$$

where $S_2 = t_1t_2 + t_2t_3 + t_3t_1$, and $S_3 = t_1t_2t_3$. Recently the image of a point T of a hyperbola with respect to the sides of an inscribed triangle has been defined in terms of slant lines [5]. In this case the line segment TT_1 was bisected by the side of the triangle. Again the three points so obtained were shown to be collinear. By recognizing that the perpendicular lines of the first paper and the slant lines of the second paper are conjugate lines of the sides of the triangle, it is

possible to generalize the theorems to a triangle inscribed in any central conic. Thus a well known theorem can be stated: *Lines of images of two diametrically opposite points of a conic with respect to the sides of an inscribed triangle are conjugate lines of the conic.*

8. The generalized Simson line. Since the line segments TT_1, TT_2, TT_3 are bisected by the corresponding sides of the triangle it follows that the feet of these lines from T are collinear. If the conic is a circle we have the Simson line. Otherwise a generalized Simson line.

9. Conclusion. The above theory applies to any central conic. The author's 1941 article [7] applied only to a triangle inscribed in a rectangular hyperbola. The wider generalization is obtained by using the concept of conjugate lines instead of that of anti-parallel lines which were defined in terms of the asymptotes of the hyperbola.

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THE NOTS REAC

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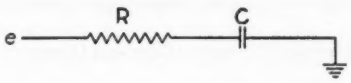
In October, 1951, the Naval Ordnance Test Station at China Lake put into operation a modest REAC installation. A REAC, by the way, is a Reeves electronic analog computer. It is probable that most mathematicians are more familiar with digital computers, for they are much more common. The digital machines are essentially arithmetical, performing very rapidly the elementary operations of arithmetic. When it comes to calculus operations, digital machines proceed by small steps, solving difference equations rather than differential equations. Computers of this type are capable of an accuracy limited only by the number of digits carried through a problem. Since this number is usually very large, extreme accuracy can be obtained. In problems such as the inversion

of matrices of higher order, such accuracy is essential. On the other hand, on digital machines the investigation of the variation of the parameters involved in a given problem is usually time-consuming and not too simple.

The REAC, like many other electronic analog computers, was designed to investigate those problems in which slide rule accuracy is sufficient and in which flexibility in the variation of parameters is important. Specifically, the REAC is a machine made to solve systems of ordinary differential equations, linear or non-linear. In modern research one encounters a great variety of problems that can be set up as a system of such equations, and frequently the initial conditions and parameters involved are known to only two or three significant figures. Once such a problem is plugged into the REAC, a solution can be plotted in much less than a minute. A parameter can be changed in a few seconds by the turn of a knob, and a new solution can be obtained shortly.

Such a machine must be able to perform integrations, algebraic additions, multiplications, and divisions. Also, it must be able to generate functions determined empirically and must be able to produce solutions as plotted graphs, sets of cards, or meter readings. In the REAC, a variable is represented as an electric potential whose magnitude and sign vary as the value of the variable. A very well regulated ± 100 volt power supply provides the frame of reference for all calculations.

Let us consider the problem of integration. Suppose a fixed voltage, e, were to be applied to the following resistor-condenser network.

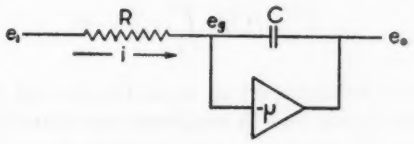


The applied voltage, e, will cause a current to pass through the resistor, R, and to accumulate as a charge on the condenser, C. The effect of this charge will be to oppose further flow of current. As a consequence, the potential across C will start to rise and will approach the value e asymptotically. At first, when the opposing voltage across the condenser is small, the rise in this voltage will be nearly linear and approximately proportional to the applied voltage. Thus, for a short time the condenser acts like an integrator. If the effect of the opposing voltage across C could be eliminated, the current stored in C would continue to rise at a rate proportional to the applied voltage. This desideratum is achieved in the REAC by means of a direct current amplifier having very high gain and using negative feedback.

Such an amplifier will be represented by the triangle,

where $-\mu$ denotes the gain. In the REAC, μ is of the order of 10^7. Let us place

such an amplifier in parallel with the condenser, C, as follows.



With an applied voltage e_i, let the resulting current be denoted by i. Since the amplifier draws no current (its input is applied to the grid of a tube), all of the current will be stored in the capacitor. In the computer, the amplifier works in such a manner that any current drawn from the output of the circuit above will be supplied by the amplifier. We shall let e_g denote the voltage at the junction indicated. If R is measured in ohms and C in farads, then we have, by Ohm's Law,

$$i = (e_i - e_g)/R.$$

Also,

$$Q = C(e_g - e_0),$$

where Q is the charge in coulombs on the condenser. Also,

$$Q = \int_0^t i\,dt + Q_0,$$

Q_0 being the charge on the capacitor at time $t=0$. Thus,

$$C(e_g - e_0) = \int_0^t (e_i - e_g)/R\,dt + Q_0.$$

Also, $e_0 = -\mu e_g$, so that

$$e_0 + e_0/\mu = -(1/RC)\int_0^t e_i dt - (1/\mu RC)\int_0^t e_0 dt - Q_0/C.$$

If μ is very large, $e_0 \doteq -(1/RC)\int_0^t e_i dt - Q_0/C$. In particular, if R is one megohm and C is one microfarad,

$$e_0 \doteq -\int_0^t e_i dt - V_0.$$

The quantity V_0, the voltage across C at time $t=0$, is available as an initial condition for the integration. It may be noted that the effect of the amplifier is to make e_g a virtual ground, for $e_g = -e_0/\mu \doteq 0$.

In the REAC at NOTS, the capacitors are fixed at 1 microfarad, but the input resistors are available in a variety of sizes from 0.01 to 10 megohms. A moment's consideration will show that if the input resistor is R megohms and

C is 1 microfarad, then

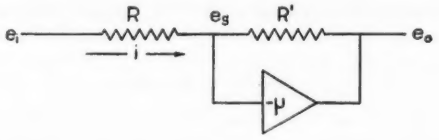
$$e_0 \doteq -(1/R)\int_0^t e_i dt - V_0.$$

Furthermore, if several voltages, say e_1, e_2, and e_3, are fed into the capacitor C through resistances of R_1, R_2, and R_3 megohms respectively, then

$$-e_0 \doteq \int_0^t (e_1/R_1 + e_2/R_2 + e_3/R_3)dt + V_0.$$

Thus, the integration of a sum is possible in a single step.

If the condenser C were replaced by a resistor, as below,



a different operation would be performed.

Since the current through R and R' is the same,

$$i = (e_i - e_g)/R = (e_g - e_0)/R'.$$

Since $e_0 = -\mu e_g$, we have

$$(e_i + e_0/\mu)/R = -(e_0/\mu + e_0)/R',$$

or

$$R'e_i + R'e_0/\mu + Re_0/\mu + Re_0 = 0,$$

or

$$e_0 = -R'e_i/(R + R'/\mu + R/\mu) \doteq -e_i(R'/R).$$

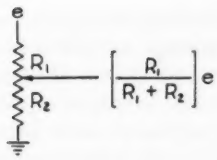
Again, the last approximation is very good because of the very large magnitude of μ.

Similarly, if we have several inputs, say e_1, e_2, and e_3 fed in through resistors R_1, R_2, and R_3, we find that

$$e_0 \doteq -[(R'/R_1)e_1 + (R'/R_2)e_2 + (R'/R_3)e_3].$$

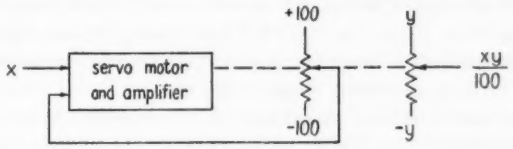
Thus, such a network can be used to add several variables and to manipulate their scale factors, or it can be used as a simple sign changer.

In the REAC, only certain values of resistance are available for such input scaling. Hence, it becomes necessary to have a way of making finer scale changes. This can be done by the following simple device, utilizing a 30,000 ohm potentiometer.



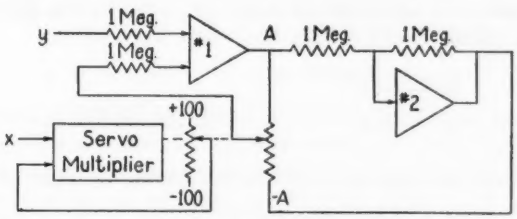
Thus, if a voltage e is fed to the top of the potentiometer, any proper fraction of e can be taken off to be used elsewhere in the computer.

Multiplication is done by means of servomechanisms as illustrated below.



When x is fed into the servo amplifier, the servo motor seeks a point on the follow-up potentiometer (hereafter called "pot") that has a voltage x. The motor shaft is then positioned so that all of the pots driven by the shaft are $x/100$ of the distance from the center to one end. If a second signal, y, is put on the upper end of such a multiplying pot and $-y$ is put on the bottom, the resulting voltage tapped off will be $xy/100$.

Division is accomplished by means of a servo multiplier and an amplifier connected as below.



Since x is fed into the servo, the pots are positioned at $x/100$ of the way from the center to the top. If we designate by A the output of amplifier 1, then the voltage tapped from the second pot will be $xA/100$. Since this voltage and y are fed into amplifier 1 through equal resistors, and the gain of the amplifier is $-\mu$, we have

$$-\mu(y + xA/100) = A, \quad \text{or} \quad y + xA/100 = -A/\mu.$$

Since μ is extremely large in comparison with A, this relationship becomes

$$y + xA/100 \doteq 0, \quad \text{or} \quad A \doteq -100y/x.$$

The REAC also has provision for the introduction of empirical functions. One such device is the electronic function generator that utilizes lantern slides.

A beam from a cathode ray tube rides on top of a lantern slide curve and is sensed by a photoelectric cell. The voltage output is proportional to the vertical deflection of the beam. Another input device consists of a servo-operated rotating drum on which an arbitrary function can be represented by a fine wire glued to the surface. A resistor card riding on the wire permits the functional voltage to be picked off for use elsewhere in the computer.

Output devices consist of a plotting table on which any variable can be plotted against any other, and a four-channel hot wire recorder on which any four variables can be plotted against time. In addition, three of the rotating drums described as input devices can also be used to plot any variable against any other. Altogether, then, the NOTS REAC can plot out eight variables at a time. If more are needed, it becomes necessary to replug several wires and re-run the problem, a process requiring about a minute.

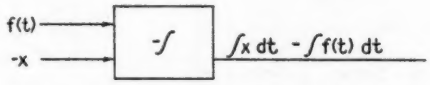
As an illustration of REAC operation, let us consider the second order linear differential equation:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = f(t).$$

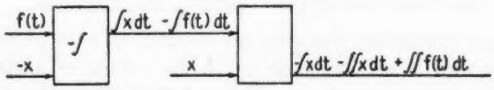
This we shall rewrite as

$$x = -\int x dt - \int\int x dt + \int\int f(t) dt.$$

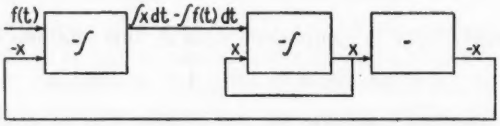
We shall suppose that the forcing function, $f(t)$, is available. In addition, suppose we had the negative of our solution, $-x$. Let us feed these into an integrator, each with scale factor 1.



Let us feed this output, together with x, into another integrator.



We observe that the output is precisely x. Then, if we feed x through an inverting amplifier with a gain of -1, we have both the x and $-x$ required to feed into the two integrators.



In order for the problem to be meaningful, initial conditions must be specified.

These initial conditions are introduced by charging the capacitors of the two integrators prior to starting the solution. The starting mechanism is merely a switch that simultaneously removes the voltage sources applied to the capacitors and connects the inputs to the amplifiers. The solution, x, can now be taken from the second integrator and plotted against t, which can be obtained by feeding a small negative voltage into a one-megohm input of an integrator. The value of the small voltage depends only on the scale desired and the length of time required for the solution.

The installation at NOTS has, at present, 14 integrators. That is, it is possible to solve seven simultaneous second-order differential equations, not necessarily linear.

Although there are about 300 REAC facilities in this country, it is doubtful if any two are alike. They are a bit like the old Model T Ford in that the original machine is generally regarded merely as a point of departure. It is a challenge to the operator's ingenuity, and modifications usually accumulate rapidly. RAND Corporation of Santa Monica, California, has probably been the source of more extensive alterations than anyone, and many of these modifications have now been adopted by Reeves.

SOME INTERESTING SERIES RESULTING FROM A CERTAIN MACLAURIN EXPANSION

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The expansion of $(\arcsin t)^2$ into a Maclaurin Series can be quite a tedious calculation since it is necessary to obtain $[(d^n/dt^n)(\arcsin t)^2]_{t=0}$. The approach in the present paper—although somewhat roundabout—does give the Maclaurin series a very simple form and we shall in addition be able to arrive at several interesting "by-products."

We consider

$$(1) \qquad I = \int_0^{\pi/2} \frac{dx}{1 - t\cos x} \qquad |t| < 1.$$

This integral may be evaluated in the following manner.

$$I = \int_0^{\pi/2} \frac{1 + t\cos x}{1 - t^2\cos^2 x}\,dx = \int_0^{\pi/2} \frac{dx}{1 - t^2\cos^2 x} + \int_0^{\pi/2} \frac{t\cos x\,dx}{1 - t^2\cos^2 x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x\,dx}{1 - t^2 + \tan^2 x} + \int_0^{\pi/2} \frac{t\cos x\,dx}{1 - t^2 + t^2\sin^2 x}$$

$$= \frac{1}{\sqrt{1-t^2}}\left[\arctan\left(\frac{\tan x}{\sqrt{1-t^2}}\right) + \arctan\left(\frac{t\sin x}{\sqrt{1-t^2}}\right)\right]\Bigg|_0^{\pi/2}$$

$$= \frac{\pi}{2\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}}\arctan\left(\frac{t}{\sqrt{1-t^2}}\right).$$

Thus we have

$$I = \frac{\pi}{2\sqrt{1-t^2}} + \frac{\arcsin t}{\sqrt{1-t^2}}. \tag{2}$$

Consider (1) once again. We may write

$$I = \int_0^{\pi/2} \{1 + t\cos x + t^2\cos^2 x + \cdots\}\,dx$$

or

$$I = \frac{\pi}{2}\left\{1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \cdots\right\} + t\left\{1 + \frac{2}{1\cdot 3}t^2 + \frac{2\cdot 4}{1\cdot 3\cdot 5}t^4 + \cdots\right\} \tag{3}$$

since

$$\int_0^{\pi/2} \cos^{2p} x\,dx = \frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots(2p)}\,\frac{\pi}{2} \tag{4}$$

and

$$\int_0^{\pi/2} \cos^{2p-1} x\,dx = \frac{2\cdot 4\cdots(2p-2)}{1\cdot 3\cdots(2p-1)}. \tag{5}$$

Noting that

$$\frac{\pi}{2\sqrt{1-t^2}} = \frac{\pi}{2}\left\{1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \cdots\right\}$$

we have upon comparison of (2) and (3) the result

$$\frac{\arcsin t}{\sqrt{1-t^2}} = t + \frac{2}{1\cdot 3}t^3 + \frac{2\cdot 4}{1\cdot 3\cdot 5}t^5 + \cdots. \tag{6}$$

Integration of (6) from 0 to t, yields the result

$$(\arcsin t)^2 = t^2 + \frac{2}{1\cdot 3}\left(\frac{t^4}{2}\right) + \frac{2\cdot 4}{1\cdot 3\cdot 5}\left(\frac{t^6}{3}\right) + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\left(\frac{t^8}{4}\right) + \cdots \tag{7}$$

which is the required Maclaurin expansion.

It follows from (7) that

$$\frac{d^n}{dt^n}(\arcsin t)^2\Big|_{t=0} = \frac{(2p)!}{p}\left(\frac{2\cdot 4\cdots(2p-2)}{1\cdot 3\cdots(2p-1)}\right) = 2[2\cdot 4\cdots(2p-2)]^2 \tag{8}$$

if $n=2p$, and zero otherwise.

From (7), upon letting $t=1$ we obtain the interesting result

$$\frac{\pi^2}{4} = 1+\frac{2}{1\cdot 3}\left(\frac{1}{2}\right)+\frac{2\cdot 4}{1\cdot 3\cdot 5}\left(\frac{1}{3}\right)+\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\left(\frac{1}{4}\right)+\cdots. \tag{9}$$

Another interesting result, but much better known, may be obtained as follows.

Let $t=\sin\theta$ in (7) so that we obtain

$$\theta^2 = \sin^2\theta+\frac{2}{1\cdot 3}\frac{\sin^4\theta}{2}+\frac{2\cdot 4}{1\cdot 3\cdot 5}\frac{\sin^6\theta}{3}+\cdots. \tag{10}$$

Integrating from 0 to $\pi/2$, using the result

$$\int_0^{\pi/2}\sin^{2p}\theta\,d\theta = \frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots((2p)}\,\frac{\pi}{2}, \tag{11}$$

and making the appropriate simplifications we have

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots.* \tag{12}$$

Still another interesting result may be obtained as follows.

We have

$$\arcsin t = \sum_{p=0}^{\infty}\frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots(2p)}\,\frac{t^{2p+1}}{2p+1} \tag{13}$$

where the term corresponding to $p=0$ is interpreted to be equal to t. Squaring (13) we have

$$(\arcsin t)^2 = \sum_{p,q=0}^{\infty}\left\{\frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots(2p)}\right\}\left\{\frac{1\cdot 3\cdots(2q-1)}{2\cdot 4\cdots(2q)}\right\}\frac{t^{2p+2q+2}}{(2p+1)(2q+1)}. \tag{14}$$

* The author wishes to thank the referee for pointing out that the method used here was the method whereby the series in (12) was summed for the first time by Euler. At present it is, of course, considered easier to use Fourier Series.

Thus

$$(\arcsin t)^2 = \sum_{n=0}^{\infty} a_{2n} t^{2n} \tag{15}$$

where

$$a_{2n} = \sum_{p=0}^{n} \left\{\frac{1\cdot 3\cdots(2p-1)}{2\cdot 4\cdots(2p)}\right\}\left\{\frac{1\cdot 3\cdots(2n-2p-3)}{2\cdot 4\cdots(2n-2p-2)}\right\}\cdot \frac{1}{(2p+1)(2n-2p-1)} \tag{16}$$

or

$$a_{2n} = \frac{1}{2^{2n-2}} \sum_{p=0}^{n} \binom{2p}{p}\binom{2n-2p-2}{n-p-1} \frac{1}{(2p+1)(2n-2p-1)}. \tag{17}$$

Comparison of (7), (14) and (17), yields the result

$$\sum_{p=0}^{n} \binom{2p}{p}\binom{2n-2p-2}{n-p-1} \frac{1}{(2p+1)(2n-2p-1)} = \left\{\frac{2\cdot 4\cdots(2n-2)}{1\cdot 3\cdots(2n-1)}\right\} \frac{2^{2n-2}}{n}. \tag{18}$$

Several other series of interest may be summed. Multiplication of (10) by $\sin\theta$, integrating from 0 to $\pi/2$ and making use of

$$\int_0^{\pi/2} \sin^{2p+1}\theta d\theta = \frac{2\cdot 4\cdots(2p)}{1\cdot 3\cdots(2p+1)}, \tag{19}$$

we find

$$\frac{\pi}{2} = \frac{4}{3} + \left(\frac{2}{1\cdot 3}\right)^2 \frac{1}{5} + \left(\frac{2\cdot 4}{1\cdot 3\cdot 5}\right)^2 \frac{1}{7} + \left(\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7}\right)^2 \frac{1}{9} + \cdots. \tag{20}$$

Multiplication of (6) by t and integrating from 0 to t, we have

$$t - (\arcsin t)(\sqrt{1-t^2}) = \frac{1}{3} t^3 + \frac{2}{1\cdot 3\cdot 5} t^5 + \frac{2\cdot 4}{1\cdot 3\cdot 5\cdot 7} t^7 + \cdots. \tag{21}$$

Letting $t=1$ we then have the result

$$\frac{2}{1\cdot 3\cdot 5} + \frac{2\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \cdots = \frac{2}{3}. \tag{22}$$

If we put $t=\frac{1}{2}$ in (21) we have

$$(23)\qquad \frac{2}{1\cdot3\cdot5}\left(\frac{1}{2^5}\right)+\frac{2\cdot4}{1\cdot3\cdot5\cdot7}\left(\frac{1}{2^7}\right)+\frac{2\cdot4\cdot6}{1\cdot3\cdot5\cdot7\cdot9}\left(\frac{1}{2^9}\right)+\cdots=\frac{11-2\pi\sqrt{3}}{24}.$$

Letting $t=\sin\theta$ in (21) and integrating from 0 to $\pi/2$ we obtain the curious result

$$(24)\qquad \pi=4-\left\{\left(\frac{2}{1\cdot3}\right)^2+\left(\frac{2\cdot4}{1\cdot3\cdot5}\right)^2\frac{1}{2}+\left(\frac{2\cdot4\cdot6}{1\cdot3\cdot5\cdot7}\right)^2\frac{1}{3}+\cdots\right\}.$$

Another curious result may be obtained as follows. Integrating (7) from 0 to t we obtain

$$(25)\qquad t(\arcsin t)^2+2(\arcsin t)(\sqrt{1-t^2})-2t=\frac{t^3}{3}+\left(\frac{2}{1\cdot3\cdot5}\right)\frac{t^5}{2}+\left(\frac{2\cdot4}{1\cdot3\cdot5\cdot7}\right)\frac{t^7}{3}+\cdots.$$

Letting $t=\sin\theta$ in (25), integrating from 0 to $\pi/2$ and making use of (19), we find

$$(26)\qquad \pi=3+\left\{\left(\frac{1}{1\cdot3}\right)^2+\left(\frac{2}{1\cdot3\cdot5}\right)^2+\left(\frac{2\cdot4}{1\cdot3\cdot5\cdot7}\right)^2+\left(\frac{2\cdot4\cdot6}{1\cdot3\cdot5\cdot7\cdot9}\right)^2+\cdots\right\}.$$

As a last result analogous in many respects to a recently proposed advanced problem of this MONTHLY (problem 4483, April 1952, p. 254), we note that upon placing $t=1$ in (25) we obtain after some simple alterations

$$(27)\qquad \frac{\pi^2}{2}-4=\left(\frac{2}{1\cdot3}\right)\frac{1}{1^2}+\left(\frac{2\cdot4}{1\cdot3\cdot5}\right)\frac{1}{2^2}+\left(\frac{2\cdot4\cdot6}{1\cdot3\cdot5\cdot7}\right)\frac{1}{3^2}+\cdots.$$

MATHEMATICAL NOTES

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CHARACTERISTIC ROOTS OF A SET OF MATRICES

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1. Introduction. A square matrix A of order n with complex elements may be written uniquely as

$$(1)\qquad A=H+iK$$

where H and K are Hermitian matrices. Hirsch [1] and Bromwich [2] have

shown that if the characteristic roots of H are $\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n$ and the characteristic roots of K are $\beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n$ then every characteristic root, $\lambda + i\mu$ of A is such that $\alpha_1 \leqq \lambda \leqq \alpha_n$ and $\beta_1 \leqq \mu \leqq \beta_n$. This associates with A a rectangular grid in the complex plane and states that the characteristic roots of A are all within or on this grid.

A matrix B has the same associated grid as A if, and only if,

$$B = UHU^* + iVKV^* \tag{2}$$

where U and V are unitary matrices.

It is the purpose of this note to discuss the distribution of the characteristic roots of the matrices B given by (2) when $n = 2$.

From (2) we have

$$B + (h + ik)I = U(H + hI)U^* + iV(K + kI)V^*.$$

That is, if each characteristic root of B is increased by $h + ik$, then each characteristic root of H is increased by h and each characteristic root of K is increased by k. This says, in effect, that the grid associated with B may be translated without disturbing the relative positions of the characteristic roots of B with respect to the grid. Hence, there is no loss of generality in assuming that the traces of B, H, and K are all zero. We may assume that both H and K are diagonal matrices. These assumptions are desirable for the purpose of computation.

2. Characteristic roots of second order matrices. We now write

$$H = \begin{pmatrix} -\alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} -\beta & 0 \\ 0 & \beta \end{pmatrix}.$$

The unitary matrices U and V may be written as

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{where} \quad u_1 = (x_1, x_2) \quad \text{and} \quad u_2 = \rho(\bar{x}_2, -\bar{x}_1),$$

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{where} \quad v_1 = (y_1, y_2) \quad \text{and} \quad v_2 = \sigma(\bar{y}_2, -\bar{y}_1),$$

and where ρ and σ are scalars of unit modulus. Then

$$B = UHU^* + iVKV^* = \begin{pmatrix} u_1Hu_1^* + iv_1Kv_1^* & u_1Hu_2^* + iv_1Kv_2^* \\ u_2Hu_1^* + iv_2Kv_1^* & u_2Hu_2^* + iv_2Kv_2^* \end{pmatrix}. \tag{3}$$

Since the trace of B is zero, we may write

$$B = \begin{pmatrix} a + ib & \lambda + i\mu \\ \bar{\lambda} + i\bar{\mu} & -a - ib \end{pmatrix} \tag{4}$$

where a and b are real.

Denote the characteristic roots of B by $x+iy$ and $-x-iy$. Since the product of these roots is the determinant of B, equating real and imaginary parts gives

$$x^2 - y^2 = a^2 - b^2 + \lambda\bar{\lambda} - \mu\bar{\mu}, \tag{5}$$

and

$$2xy = 2ab + \lambda\bar{\mu} + \bar{\lambda}\mu. \tag{6}$$

Using the values of a and λ as defined by (3) and (4) we have

$$a = u_1 H u_1^* = \alpha(x_2\bar{x}_2 - x_1\bar{x}_1)$$

and

$$\lambda = u_1 H u_2^* = -2\alpha\bar{\beta}x_1x_2.$$

Hence

$$\begin{aligned} a^2 + \lambda\bar{\lambda} &= \alpha^2(x_2\bar{x}_2 - x_1\bar{x}_1)^2 + 4\alpha^2 x_1x_2\bar{x}_1\bar{x}_2 \\ &= \alpha^2(x_1\bar{x}_1 + x_2\bar{x}_2)^2 = \alpha^2. \end{aligned}$$

Similarly

$$b^2 + \mu\bar{\mu} = \beta^2.$$

Thus, equation (5) becomes

$$x^2 - y^2 = \alpha^2 - \beta^2$$

which is independent of the choice of U and V. We may now state:

THEOREM 1. *If A is any 2×2 matrix with complex elements, the characteristic roots of A are the end points of a diameter of the rectangular hyperbola through the vertices of the rectangle associated with A.*

If either $\alpha=0$ or $\beta=0$ this rectangle becomes a line segment and the characteristic roots are the end points of the segment. There is a matrix B with equal roots in the set if, and only if, the rectangle is a square and in this case the characteristic roots of every matrix in the set are on the diagonals of the square.

The *field of values* of the matrix B is the ellipse inscribed in the associated rectangle and having the characteristic roots of B as foci. There is one, and only one, ellipse inscribed in the rectangle with axes in any given direction. The foci of this ellipse are on the equilateral hyperbola passing through the vertices of the rectangle. Such an ellipse is the field of values of some matrix associated with the rectangle. All matrices associated with the rectangle are given by (2). Hence the fields of values of all matrices given by (2) and the set of all ellipses inscribed in the rectangle are identical. The set of all characteristic roots of all matrices given by (2) is the set of all points in or on the rectangle and on the equilateral hyperbola passing through the vertices of the rectangle.

The field of values of B is a circle if, and only if, its two characteristic roots

are equal, and is a line segment if, and only if, its characteristic roots are opposite vertices of the rectangle.

3. Matrices for which H and K have multiple roots. We now establish the following:

THEOREM 2. *If H has the characteristic root α of multiplicity r and K has the characteristic root β of multiplicity s then every matrix B in (2) has the characteristic root $\alpha+i\beta$ of multiplicity $r+s-n$ at least.*

From (2) we have

$$B - (\alpha + i\beta)I = U(H - \alpha I)U^* + iV(K - \beta I)V^*. \tag{7}$$

The rank of $U(H-\alpha)U^*$ is $n-r$ and the rank of $V(K-\beta I)V^*$ is $n-s$ and hence the rank of B does not exceed $2n-r-s$. Hence $B-(\alpha+i\beta)I$ has the characteristic root zero of multiplicity $r+s-n$ at least and the theorem follows.

In particular if $r=s=n-1$ the equation (7) becomes

$$B - (\alpha + i\beta)I = -n\alpha L^*L - i(n\beta)M^*M \tag{8}$$

where $L=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the last row of U^* and $M=(\mu_1, \mu_2, \cdots, \mu_n)$ is the last row of V^*. We now assume that U and V are selected so that $B-(\alpha+i\beta)I$ is triangular and so that the first $n-2$ diagonal elements are zero. This may always be done. It follows that $\lambda_1=\lambda_2=\cdots=\lambda_{n-2}=\mu_1=\mu_2=\cdots=\mu_{n-2}=0$ and hence

$$B = \begin{pmatrix} (\alpha + i\beta)I_{n-2} & 0 \\ 0 & B_1 \end{pmatrix}$$

where

$$B_1 = U_1 \begin{pmatrix} \alpha & 0 \\ 0 & -(n-1)\alpha \end{pmatrix} U_1^* + iV_1 \begin{pmatrix} \beta & 0 \\ 0 & -(n-1)\beta \end{pmatrix} V_1^*.$$

Then applying Theorem 1 we have

THEOREM 3. *If the characteristic equation of H is $(x-c)(x-a)^{n-1}=0$ and the characteristic equation of K is $(x-d)(x-b)^{n-1}=0$, then every matrix B given by (2) has the characteristic root $a+ib$ of multiplicity $n-2$ and the two remaining characteristic roots are the ends of a diameter of the equilateral hyperbola through (a, b), (a, d), (c, b), and (c, d).*

References

1. Hirsch, A., Sur les racines d'une équation fondamentale, Acta Mathematica, vol. 25, 1902, pp. 367–370.

2. Bromwich, T. J. I'a., On the roots of the characteristic equation of a linear substitution, Acta Mathematica, vol. 30, 1906, pp. 295–304.

DEFINITION OF THE LEBESGUE INTEGRAL

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Many definitions of the Lebesgue integral have been given. However, the following definition, which is a simple generalization of the Cauchy integral for continuous functions on a closed interval, is not well-known in this country.*

For convenience, only real functions defined on the closed interval $[0, 1]$ will be considered, and all sets will be understood to be subsets of this interval. Suppose S is a closed set and $f(x)$ is continuous on S relative to S. Then the complement $C(S)$ of S consists of a finite or denumerable set of disjoint intervals, $I_1, I_2, \cdots, I_n, \cdots$. Let a_n and b_n be the ends points of I_n, for every $n=1, 2, \cdots$. Moreover, $f(x)$ may be extended to a continuous function $F(x)$ on $[0, 1]$ such that $F(x)=f(x)$ for $x\in S$. The Cauchy integral of $f(x)$ on S may be defined as

$$\int_S f(x)dx = \int_0^1 F(x)dx - \sum_{n=1}^{\infty} \int_{a_n}^{b_n} F(x)dx.$$

The proof that $\int_S f(x)dx$ is independent of the choice of $F(x)$, subject to the condition that $F(x)$ is a continuous extension of $f(x)$, is easy and is left to the reader.

Now, let $f(x)$ on $[0, 1]$ belong to the class $\mathcal{M}$ if there is an increasing sequence of closed sets $S_1 \subset S_2 \subset \cdots \subset S_n \subset \cdots$ such that, for every $\epsilon>0$, there is an m for which the sum of the lengths of the disjoint intervals whose union is $C(S_m)$ is less than ϵ, and such that $f(x)$ is continuous on S_n, for every n.

If $f(x)\in\mathcal{M}$ is non-negative, and $\{S_n\}$ is a sequence of closed sets having the above property relative to $f(x)$, the sequence $\{\int_{S_n} f(x)dx\}$ is non-decreasing. If this sequence converges, then $f(x)$ is summable, and the integral of $f(x)$ is defined as

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_{S_n} f(x)dx.$$

If the sequence diverges, then $f(x)$ is non-summable.

The sequence $\{S_n\}$ of closed sets corresponding to a given $f(x)\in\mathcal{M}$ is not unique. That the summability or non-summability of $f(x)$ and the value of its integral are not dependent on the choice of the closed sets is again an easy exercise which is left for the reader. The definition of summability and integral for arbitrary $f(x)\in\mathcal{M}$ may be accomplished in the usual way by writing $f(x)$ as the difference between two non-negative functions. The proof that $f(x)$, $g(x)$ summable implies $f(x)+g(x)$ summable and $\int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$ is now essentially a corollary to the corresponding theorem for continuous functions. Indeed, the whole theory of Lebesgue integration may be

* The author has been informed by Professor E. J. Mickle that a definition similar to this one is commonly used in Italy.

obtained from this definition.

Finally, although the details of the integration theory may be obtained without reference to this fact, the above class $\mathcal{M}$ is precisely the class of measurable functions. This follows by Lusin's Theorem that a function $f(x)$ on $[0, 1]$ is measurable if and only if, for every $\epsilon>0$, there is a closed set S of measure exceeding $1-\epsilon$ such that $f(x)$ is continuous on S relative to S.

A PROPERTY OF RANDOMNESS OF AN ARITHMETICAL FUNCTION

N. METROPOLIS and S. ULAM, Los Alamos Scientific Laboratory

Let $f(x)$ be a transformation of a set E into itself. One can decompose E into minimal invariant subsets or "trees" by considering for each point x the smallest set X with the following properties: 1. $x\epsilon X$, 2. If $y\epsilon X$ then $f(y)\epsilon X$, 3. If $y\epsilon X$ all the points x such that $f(x)=y$ also belong to X. One will obtain a decomposition of $E=X_1+\cdots+X_\xi+\cdots$ into disjoint trees X_ξ. These characterize $f(x)$ up to a conjugating transformation h; *i.e.*, all $h(f(h^{-1}))$ where h is an arbitrary one-one transformation of E into itself.*

Suppose that $f(x)$ is a random function defined on E with values in E, *i.e.*, for each x a value y called $f(x)$ was chosen from E, say with a uniform distribution of probability in E (assumed to be a measure space).§ In case of E finite, one could ask about the "probable" or expected number of trees in the decomposition given by f. This expected number, not easy to estimate, is likely of the order of $\log n$; n is the number of points of E. For the case where f is postulated as a one to one transformation, *i.e.*, a permutation, the trees become cycles. It is well known that the expected number of cycles is $\log n$.

We examined the number of trees for some specific *not a priori* random functions. One function, often used in various Monte Carlo problems to produce random *digits* (by iteration), is the following: let x be an integer ranging from 0 to 2^k-1 written in the binary development $x=\alpha_0 2^0+\cdots\alpha_{k-1}2^{k-1}$. Let $f(x)$ be defined as follows: we take x^2 written again in the binary notation and define $f(x)$ as the number given by the *middle* k (out of $2k$) digits of x^2. It will, of course, again range between 0 and 2^k-1.

With this $f(x)$, for $k=12$, *i.e.*, x ranging from 0 to 4095, the number of trees is 7.

Imagine that, as mentioned before, for each element x of a set E, one performs a selection of an element from E, with, say, uniform probability, *i.e.*, each time, every element has an equal chance $(=1/n)$ of being selected. One can ask of such a "random function," or transformation, for the probable number λ_0 of points y in the set E which are *not* of the form $y=f(x)$. Also one can ask for the number λ_1 of points y for which there is exactly *one* x such that $y=f(x)$—more generally, λ_i of points for which there are exactly i distinct

* See S. Ulam, Bull. Am. Math. Soc., 1943, Abstract, p. 49.

§ We consider the product space E^E. Logically, the study of a random function is, of course, the study of the probability distribution in E^E, the totality of all $f(x)$ on E into E.

values of x so that $f(x_1) = \cdots = f(x_i) = y$. For large n, the numbers λ_i should have a Poisson distribution, *i.e.*, $\lambda_i = ne^{-1}/i!$ (*e.g.*, λ_0 is the number of points not selected. But, given a number, at each choice, the probability of it *not* being selected is $1-1/n$, and this experiment is repeated n times independently—similarly for $i=1, 2, \cdots$.)

It is perhaps amusing that for some *specific* functions, for instance $f(x)$ as defined above by the "middle of the square," the actual distribution of the λ_i is extremely close to the one expected for a "random transformation." Sample values are given below:

For $k=12$; *i.e.* $n=4096$			For $k=16$, $n=65{,}536$		
i	λ_i/n observed	λ_i/n Poisson	i	λ_i/n observed	λ_i/n Poisson
0	.366	.368	0	.370	.368
1	.377	.368	1	.367	.368
2	.178	.184	2	.183	.184
3	.061	.061	3	.062	.061
4	.012	.015	4	.015	.015
5	.004	.003	5	.003	.003
			6	.0006	.0006

The only sizable deviation from what one would expect in a "random" f are, of course the sets $f^{-1}(0)$ and $f^{-1}(2^{k/2})$.

The problem of enumeration for the case $k=12$ was done with the aid of a desk calculator. The larger problem of $k=16$, $n=65{,}536$ was performed on the recently completed Los Alamos electronic computer. The computing problem consisted of generating the particular set $f(E)$ and of counting how many times each y occurred.

It was possible to keep track in the "memory" of 4096 numbers at any one time, so that it was necessary to regenerate the whole set E sixteen times to get a complete enumeration and check. The total computing time was one and one-half hours.

We propose sometime to examine, empirically on the computer, certain other arithmetical functions for their "tree" distribution and other combinatorial properties. We express our thanks to Miss Lois Cook for aid in doing both the hand computation and "coding" for the automatic calculation.

A NOTE ON ORTHOGONAL MATRICES

L. Carlitz, Duke University

1. J. L. Brenner [1] showed that the only 2×2 orthogonal matrices with elements in $GF[p, x]$ are constant matrices for $p>2$, while for $p=2$ they are of the form

$$\begin{pmatrix} 1+P(x) & P(x) \\ P(x) & 1+P(x) \end{pmatrix} \qquad (P(x) \epsilon GF[2, x]).$$

The result followed as a corollary of a theorem on polynomial parametrization. It is not difficult to give a direct proof.

In the present note we construct some special $p \times p$ orthogonal matrices with polynomial elements. Consider first the special matrix

$$S = (i - j) \qquad (i, j = 1, \cdots, p).$$

Then it is clear that $S^2 = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{p} (i - k)(k - j) = -\sum_{k=1}^{p} (k^2 - (i + j)k + ij),$$

from which it follows that

$$c_{ij} \equiv \begin{cases} -1 \pmod{p} & (p = 3) \\ 0 \pmod{p} & (p > 3). \end{cases}$$

In other words $S^2 = 0$ for $p > 3$; for $p = 3$ it is easily verified that $S^3 = 0$.

Let A' denote the transpose of the matrix A. Then $S' = -S$ and

$$(I + xS)(I + xS)' = (I + xS)(I - xS) = I$$

for $p > 3$, so that $I + xS$ is orthogonal for $p > 3$. For $p = 3$, put $A = I + xS - x^2S^2$; then

$$AA' = (I + xS - x^2S^2)(I - xS - x^2S^2) = I,$$

so that A is orthogonal.

2. The above result can be extended by considering the matrices

$$S_r = ((i - j)^r) \qquad (i, j = 1, \cdots, p; r = 0, 1, 2, \cdots).$$

Then $S_r' = (-1)^r S_r$ so that S_r is skew-symmetric for r odd. (Note that for $r = \frac{1}{2}(p-1)$, $S_r = ((i-j/p))$, where (a/p) is the Legendre symbol). Now

$$S_r S_s = (a_{ij}), \qquad a_{ij} = \sum_{k=1}^{p} (i - k)^r (k - j)^s.$$

Using the well-known formula

$$\sum_{k=1}^{p} k^r \equiv \begin{cases} 0 & (p - 1 \nmid r) \\ -1 & (p - 1 \mid r) \end{cases} \qquad (r > 0),$$

we see that

$$a_{ij} \equiv \begin{cases} 0 & (r + s < p - 1) \\ -1 & (r + s = p - 1). \end{cases}$$

Thus it follows that

$$S_r S_s = \begin{cases} 0 & (r+s < p-1) \\ -S_0 & (r+s = p-1); \end{cases} \tag{1}$$

we remark that all the elements of $S_0 = 1$.

Now consider the matrix

$$A = I + x_1 S_1 + \cdots + x_k S_{2k-1},$$

where $k < \frac{1}{2}(p-1)$ and $x_1, \cdots, x_k$ are indeterminates. It follows at once from (1) that

$$AA' = (I + x_1 S_1 + \cdots + x_k S_{2k-1})(I - x_1 S_1 - \cdots - x_k S_{2k-1}) = I,$$

so that A is orthogonal.

If $p = 4m-1$, we put

$$B = I + x_1 S_1 + \cdots + x_m S_{2m-1} + \tfrac{1}{2} x_m^2 S_0;$$

then by the second of (1)

$$BB' = I - x_m^2 (S_{2m-1}^2 + S_0) = I,$$

so that B is orthogonal. Additional examples are easily constructed.

Reference

1. J. L. Brenner, Polynomial parametrizations, this MONTHLY, vol. 58, 1951, pp. 327–329.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE DERIVATIVE OF cos x

A. G. ANDERSON, Oberlin College

The derivation of the expression

$$\frac{d}{dx} \cos x = -\sin x \tag{1}$$

in most elementary calculus texts is based on the identity $\cos x = \sin(\frac{1}{2}\pi - x)$. The purpose of the present note is to advocate an alternative derivation based on the fundamental identity $\sin^2 x + \cos^2 x = 1$.

Let $\sin x = u$ and $\cos x = v$. Then $u^2 + v^2 = 1$. Differentiation of this expression

with respect to x yields

$$2u\frac{du}{dx}+2v\frac{dv}{dx}=0.$$

Thus

$$\frac{dv}{dx}=-\frac{u}{v}\frac{du}{dx},$$

and (1) is seen to hold.

There is of course nothing in this presentation which is unknown to any reader, but it is my feeling that the method is often overlooked. The derivation given here can be presented without modification either in an ordinary calculus class or a freshman survey course including the elements of calculus. It has been my experience that students, particularly those with sketchy backgrounds in trigonometry, grasp this derivation more readily and that it is more useful in serving as a reminder that basic techniques are applicable in the case of trigonometric functions as well as for the algebraic functions in connection with which they were first obtained.

GEOMETRIC CONVERGENCE

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In discussing the convergence of a real sequence of the form $u_{n+1}=F(u_n)$, it is often useful to look at the geometry associated with such a sequence.

Let us first consider the sequence $u_{n+1}=\sqrt{k+u_n}$. We wish to determine the values of k and u_0 such that the sequence $\{u_n\}$ converges, and also to find its limit.

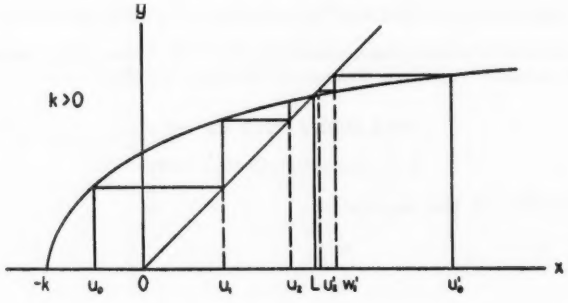


FIG. 1

We first graph $y=\sqrt{k+x}$ and $y=x$ (see Fig. 1), and then construct a ladder starting from $x=u_0$ between the parabola $y=\sqrt{k+x}$ and the straight line $y=x$

by horizontal and vertical displacements as shown. If the ladder tends to a point, then the sequence is convergent and will approach this point as a limit. This limit point will be one of the intersections points of $y=x$ and $y=\sqrt{k+x}$ in case there is more than one. Since both the parabola and line are monotonic increasing curves, it is evident from the plot that for $-k<u_0<L$, the sequence $\{u_n\}$ is monotonically increasing and bounded. Similarly, for $u_0 \geqq L$, the sequence is monotonically decreasing and bounded. Thus in either case, the sequence approaches L as a limit. For $u_0<-k$, we do not get a real sequence. These results can easily be verified analytically.

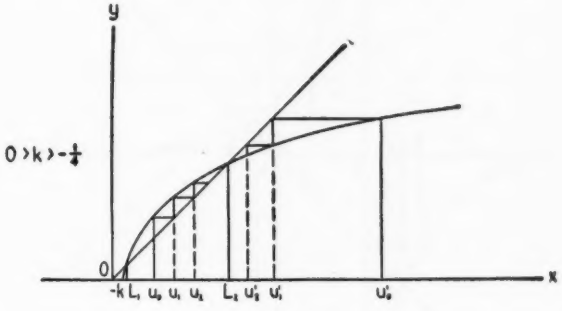


FIG. 2

For $0>k>-1/4$, the graph is given by Figure 2. If $u_0 \geqq L_2$, $\{u_n\}$ is a monotonic decreasing sequence approaching L_2 as a limit. For $L_1<u_0<L_2$, $\{u_n\}$ is monotonically increasing and also approaches L_2. For $u_0<L_1$, u_n is eventually complex.

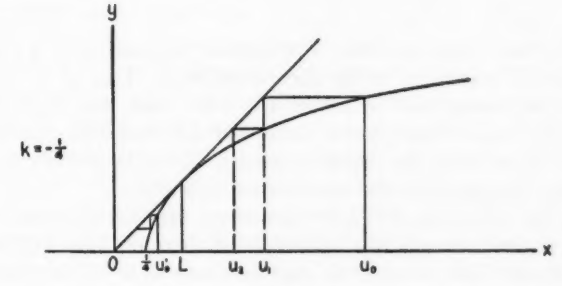


FIG. 3

When $k=-1/4$, the line $y=x$ is tangent to the parabola (see Fig. 3). If $u_0 \geqq 1/2$, the sequence converges. If $u_0<1/2$, the sequence is eventually complex. The value of $k=-1/4$ is obtained by making $x=\sqrt{k+x}$ have a double root.

Finally, for $k<-1/4$, the parabola doesn't intersect the straight line $y=x$.

This leads again to a complex sequence.

A more involved problem is given by the sequence $u_{n+1}=1/(2+u_n)$. We first plot $y=1/(2+x)$ and $y=x$ (see Fig. 4).

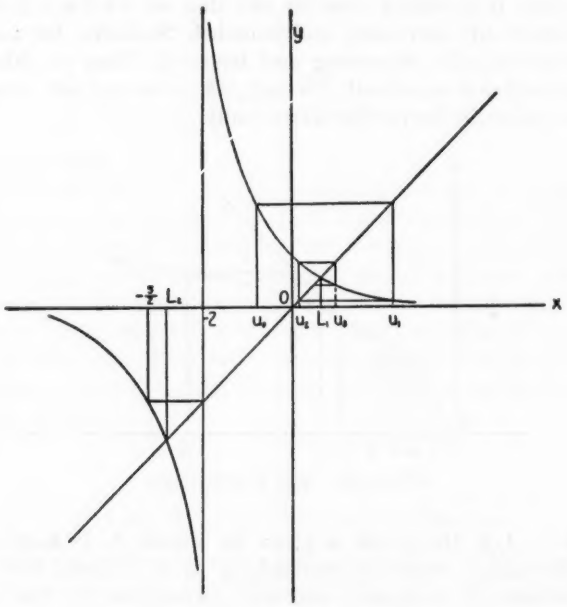


FIG. 4

If $u_0>-2$, then $\{u_{2n}\}$ is either a monotonic increasing or a monotonic decreasing bounded sequence, while correspondingly, $\{u_{2n-1}\}$ is monotonically decreasing or increasing and bounded. In either case, the sequence $\{u_n\}$ approaches L_1 as a limit. This gives a picture of a rectangular converging spiral. If $u_0<-5/2$, we go from the negative branch of the hyperbola to the positive branch and the discussion is the same as for $u_0>-2$.

However, for $-5/2<u_0<-2$, the sequence may or may not converge. If any $u_n=-2$, the sequence is not defined from there on and will be considered as non-convergent. The values of u_0 that lead to a $u_n=-2$ can be obtained by considering the inverse sequence $\{a_n\}$ satisfying the equation $a_n=1/(2+a_{n+1})$ with $a_0=-2$. Then if $u_0=a_r$, $u_r=-2$. Thus if $u_0\neq a_r$ in the given range, the spiral will first unwind about L_2 and then eventually wind around and converge to L_1. Incidentally, we can make the unwinding process take as many steps as desired by taking u_0 sufficiently close to L_2.

In the case $u_r=-2$ the sequence will converge if we define $u_{r+2}=0$.

An example from the 1952 William Lowell Putnam Competition lends itself readily to the type of geometric analysis illustrated. This was to show that $\lim u_n$, (where $u_{n+1}=\cos u_n$) is independent of u_0 (u_0 real) (see Fig. 5).

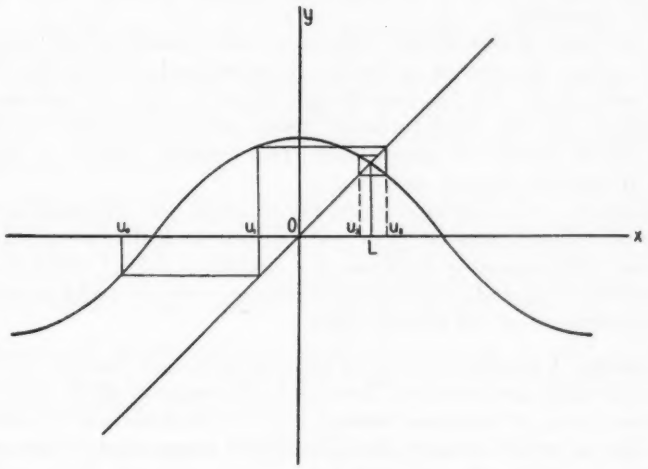


FIG. 5

ON THE DEFINITION OF FUNCTIONS*

H. P. THIELMAN, Iowa State College

1. Fundamental concepts. We shall begin with certain undefined concepts such as set and element of a set. We might give some synonyms of these terms, but the synonyms would then have to be left undefined. Thus a set might be described as a collection of definite, distinct objects associated in thought. By the word "definite" we mean here the following. Given a set S and an object, one and only one of the next two statements is true: the object is an element of the given set; the object is not an element of the given set. It is not required that we, nor anyone else, need to know which of these statements is true. Thus the collection of ladies present that are over sixteen years of age at this moment is a set, even though we or anybody may not know whether a given lady belongs to this set or not. The word "distinct" is used here to indicate that no object can be considered as an element of a given set more than once. For example, the collection of letters a, b, a, c, b constitutes the set a, b, c. In contradistinction to the expression "set of elements" (or set of sets) we shall use the terms "collection of elements" (or collection of sets) to indicate that it is not

* An excerpt from an address presented under the title *Types of Functions* to the Minnesota Section of the Mathematical Association of America at the invitation of the Executive Committee on May 10, 1952. Another excerpt from this address is published in the March issue of this MONTHLY.

implied that the elements (or the sets) be all distinct.

When a symbol such as a letter, say x, stands for an unspecified element of a set, then this symbol x is said to *vary* over the set, and the symbol x is called a *variable* on the set. If x varies over a set which consists of only one element then x is called a constant.

An *ordered pair of elements* is a collection which consists of two elements one of which has been designated as the first. If an ordered pair consists of the elements a and b of which a has been designated as the first, we indicate this by the symbol (a, b). The word "ordered" refers only to the order in which the elements appear within the parentheses. Two ordered pairs (a, b) and (c, d) are equal if and only if $a=c$, and $b=d$.

A *relation* is a set of ordered pairs. For example, the relation $>$ (greater) is the set of all ordered pairs (x, y) of real numbers such that $x-y$ is a positive real number. The *domain of definition of a relation* is the set which consists of all the first elements, and the *range of the relation* is the set which consists of all the second elements of the ordered pairs.

2. Function. A *function* is a set of ordered pairs such that no two ordered pairs have the same first element. The set which consists of all the first elements of the ordered pairs of the given function is called the *domain of definition of the function*. The set which consists of all the second elements of the ordered pairs is called the *range of the function*.

Let f be a given function with domain of definition X, and with range Y. If x stands for an unspecified element of X, x is called the *independent variable* of the given function. If y stands for an unspecified element of the range Y, then y is called the *dependent variable* of the given function. For a given ordered pair (x, y) of f, y is called the *image* of x under f, while x is called the *counter image* or *source* of y under f. The image of x under f is also called the *value of the function f at x*, and is denoted by $f(x)$. A function f whose domain of definition is X, and whose range is Y is frequently denoted by $f: X \rightarrow Y$, and is referred to as a function *on X onto Y*. If X is a subset of X_1, and Y is a subset of Y_1, then f is said to be a function *from X_1 to Y_1*, or *from X_1 into Y_1*.

3. Inverse function. Let f be a function on X onto Y. Thus f is a set of ordered pairs (x, y) where x is an element of X, and y is an element of Y. If f is such that no two of its ordered pairs have the same second element, then the function obtained from f by interchanging in each ordered pair (x, y) the places of x and y is called the *inverse function* of f. This function is indicated by f^{-1}. Its domain of definition is Y, and its range is X.

The symbols $f(x)$, $f^{-1}(y)$, or $y=f(x)$, $x=f^{-1}(y)$ should be used in referring to a function and its inverse only when it is clear from the context what the domains of definition and the ranges of these functions are.

4. Inverse relations. Let f be a function on X onto Y. If f is such that two or more of its ordered pairs have the same second element, then the set of ordered pairs obtained from f by interchanging in each ordered pair (x, y) the places of

y and x does not constitute a function from Y to X, but it does constitute a relation. The symbol $\{f^{-1}(y)\}$ can be used to denote the set which consists of all those elements of X which are counter images of y under f. Thus for every y in the range Y of f, $\{f^{-1}(y)\}$ represents a subset of X. If the inverse function f^{-1} exists, then for each y of Y, $\{f^{-1}(y)\}$ represents the set which consists of the unique element $f^{-1}(y)$.

The definition of function leaves the nature of the sets which constitute the domain of definition and the range of a function unspecified. These sets may be sets of sets, sets of real or complex numbers, sets whose elements are vectors or other arbitrary sets.

If the range of a function is a subset of the set of real numbers, the function is called a *real function*, if the domain of definition of a function is a subset of the real numbers we have a function of a *real variable*. If the domain of definition of a function is a set of sets, we have a *set function*. A set function whose range is a subset of the set of real numbers is called a *real set function*.

5. One-to-one correspondence. If f is a function on X onto Y which has an inverse f^{-1} on Y onto X, then f and f^{-1} are said to constitute a *one-to-one correspondence* between X and Y.

6. Unrestricted functions. Let f be a real function of a real variable. No other restrictions are put on this function. What can be said of such a function? This question has been partially answered in recent years, and the answer is quite different from what had been surmised by some mathematicians. Even E. W. Hobson in his 1921 edition of the Theory of Functions of Real Variables wrote: "No elaborate theory is required for functions which retain their complete generality, . . . since few deductions of importance can be made from the definition which will be valid for all functions." This conjecture was proven to be false by the late Professor Henry Blumberg who has developed an extensive theory which reveals that every unrestricted real function of real variables possesses many properties which are far from obvious and yet have a beauty of simplicity that has attracted the interest of many mathematicians. Since this theory has been presented by its originator in a number of nontechnical addresses before the American Mathematical Society, and since these addresses have been published and are easily accessible we shall not elaborate on this very interesting topic here.*

Some of these results on unrestricted functions of real variables have very recently been extended to relations in general neighborhood spaces.†

It should perhaps be mentioned that the term multiple-valued function which is frequently met in mathematical literature means a relation, that is,

* Blumberg, Henry, Properties of unrestricted real functions, Bull. Amer. Math. Soc., vol. 32, pp. 132–148, 1926; Methods in point sets and the theory of real functions, Bull. Amer. Math. Soc., vol. 36, pp. 809–830, 1930.

† Block, H. D., and Cargal, B., Arbitrary mappings, Proc. Amer. Math. Soc., vol. 3, 1952, pp. 937–941.

it is a name for a set of ordered pairs in which two distinct ordered pairs may have the same first element. If one admits the term multiple-valued function, then the concept *function* as given in section 2 of the present paper would be described as a single-valued function. In recent years the tendency in mathematical literature has been to limit the term function so as to mean a single-valued function. An equation such as $x^2+y^2=1$, where $-1 \leqq x \leqq 1$, $-1 \leqq y \leqq 1$, does not define a function. There exist infinitely many functions whose images and counter images satisfy the condition imposed on them by the equation of the preceding sentence. Thus if $y=\sqrt{1-x^2}$ if x is a rational number, and $y=-\sqrt{1-x^2}$ if x is an irrational number, there is defined a function for which $x^2+y^2=1$. By a proper reassignment of the values of y for given values of x, arbitrarily many functions can be constructed whose images and counter images satisfy the given equation in x and y.

It is gratifying to see that the unambiguous definition of a function, which is advocated here, has been included in a new elementary textbook.‡

‡ Randolph, John F., Calculus, p. 10, Macmillan Co., 1952.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1061. *Proposed by Walter Penney, Washington, D.C.*

Solve for n, given that the equation

$$\sum_{i=1}^{n} (x+i-1)(x+i) = 10n$$

has roots r and $r+1$.

E 1062. *Proposed by Leo Moser, University of Alberta*

(1) Find six positive integers, not exceeding 24, such that the sums of the numbers in the possible subsets of those numbers will all be different.

(2) Prove that no seven positive integers, not exceeding 24, can have sums of all subsets distinct.

E 1063. *Proposed by J. V. Whittaker, U.C.L.A.*

Show that if $a \geqq 2$ and $x > 0$, then $a^x + a^{1/x} \leqq a^{x+1/x}$, equality holding if, and only if, $a = 2$ and $x = 1$.

E 1064. *Proposed by Jacob Samoloff and Albert Wilansky, Lehigh University*

Let $f(x)$ be continuous and $f'(x)$ exist in a neighborhood of $x = c$. Suppose that there exists a continuous function $\theta(h)$, with $0 < \theta(h) < 1$, satisfying the equation

$$f(c + h) - f(c) = hf'[c + h\theta(h)].$$

Does it follow that $f'(x)$ is continuous at $x = c$?

E 1065. *Proposed by C. S. Ogilvy, Syracuse University*

Find the largest plane section of a given solid right circular cylinder.

SOLUTIONS

The Game of Nim

E 1031 [1952, 551]. *Proposed by A. J. Friedland, Brookhaven National Laboratory, Upton, L.I.*

A certain game is played between two participants in the following manner. A set of numbers is written down and the players alternately cross out any one of the numbers and substitute any smaller number until nothing remains but zeros. The player putting down the last zero is the winner. What method of play should one follow in order to win this game?

Solution by C. A. Swanson, University of British Columbia. The game is nothing but *Nim* in disguise. A complete treatment is given by C. L. Bouton, *Annals of Mathematics*, Series 2, Vol. III, pp. 35–39 (October 1901). To win the game from an uninitiated opponent, one writes all the numbers in the binary scale, adds the coefficients of each power of two so obtained, and at each play reduces one of the given numbers so that the sum of the coefficients of each power of two becomes an even number.

There is a machine in the Department of Physics at the University of British Columbia which plays this game for a particular set of given numbers. The machine not only plays to win, but it detects an opponent who tries to cheat, and then refuses to play.

Also solved by Julian Braun, Richard Courter, E. T. Frankel, Michael Goldberg, R. E. Greenwood, S. W. Hahn, B. A. Hausmann, Patricia James, John Jones, Jr., M. S. Klamkin, Octave Levenspiel, D. M. Mandelbaum, Leo Moser, W. J. Pervin, D. L. Silverman, G. W. Walker, and the proposer.

Some ready references where this game is discussed are: Ball-Coxeter, *Mathematical Recreations*, 11th ed., 1939; Uspensky and Heaslet, *Elementary Number Theory*, 1939; Hardy and Wright, *An Introduction to the Theory of*

Numbers, 1938; Northrop, *Riddles in Mathematics*, 1944; Kraitchik, *Mathematical Recreations*, 1942; Jones, *Elementary Concepts of Mathematics*, 1947.

A digital computer has been designed to play the game. See *Electronics*, XXV, no. 11, pp. 155–157 (November, 1952): "Digital Computer Plays Nim."

The MONTHLY has published a number of articles on the game of Nim. See for example, D. P. McIntyre, "A New System for Playing the Game of Nim" [1942, 44–45]; E. U. Condon, "The Nimatron" [1942, 330–332]; L. S. Recht, "The Game of Nim" [1943, 435].

Silverman proposed to study the variation wherein each player has the option of playing on n numbers, where $n>1$.

A Differential Equation

E 1032 [1952, 551]. *Proposed by E. W. Marchand, Eastman Kodak Company, Rochester, N.Y.*

Solve the differential equation $(dy/dx)^3-3y(dy/dx)^2+4y^3=a$.

Solution by C. R. Sparks, Hampton Institute. By differentiating the given equation with respect to x we obtain

$$(dy/dx)(d^2y/dx^2 - dy/dx - 2y)(dy/dx - 2y) = 0, \tag{1}$$

which must be satisfied by any solution y of the original equation. It follows that any solution y must have one of the following three forms:

$$y = C_1, \qquad y = C_2e^{2x} + C_3e^{-x}, \qquad y = C_4e^{2x}. \tag{2}$$

It is readily seen that the third form does not satisfy the original equation (unless $a=0$). By substituting each of the first two forms of (2) into the original equation we easily find, respectively,

$$y = (a/4)^{1/3}, \qquad y = (a/27K^2)e^{2x} + Ke^{-x},$$

where K is an arbitrary constant.

Also solved by A. N. Aheart, A. P. Bobbett, J. J. Corliss, H. M. Feldman, R. R. Gutzman, Sherman Kingsbury, M. S. Klamkin, W. H. McKenzie, R. V. Muguercia, Herbert Schalz, O. E. Stanaitis, F. Underwood, J. E. Wilkins, Jr., and the proposer.

The Balancing Beam

E 1033 [1952, 551]. *Proposed by G. B. Robison, University of Connecticut*

A horizontal plank is balanced on a cylindrical surface whose elements are perpendicular to the length of the plank. What is the cross section of the cylindrical surface if the plank has neutral equilibrium?

Solution by P. F. Hultquist, Boulder, Colo. With the plank in a horizontal position take the x-axis parallel to the plank and through the center of gravity, and take the y-axis vertically downward through the center of gravity and

point of support. Let the coordinates of the point of support when the plank is horizontal be $(0, a)$, and the coordinates of any other point of support be (x, y). Since the plank remains always tangent to the curve, and the center of gravity remains always on the x-axis vertically in line with the point of tangency, we can write $y = a \sec \theta$, where θ is the angle of inclination of the plank. Then

$$y = a\sqrt{1 + (dy/dx)^2},$$

whence we obtain the explicit equation,

$$y = a \cosh (x/a),$$

the equation of a *catenary*.

The solution is valid until the plank is tilted enough to cause slipping. This will occur when the component of weight in the direction of the tangent exceeds the coefficient of friction times the normal force between plank and curved surface. The value of the critical angle is $\theta = \tan^{-1} \mu$, where μ is the coefficient of friction.

Also solved by J. W. Baldwin, C. W. Bruce, Vern Hoggatt and Octave Levenspiel (jointly), M. S. Klamkin, C. S. Ogilvy, F. Underwood, and the proposer.

Four Digit Numbers and Their Reverses

E 1034 [1952, 551]. *Proposed by M. Narasimhamurthy, Presidency College, Madras, India*

Add to its own reverse the difference between any four digit number and its reverse. What are the possible results and what are the probabilities of getting each of these results?

Solution by G. W. Walker, Buffalo, N.Y. For the sake of neatness we assume that any of the digits may be 0. There are then 10^4 possible four digit numbers. Let the original number be $1000a+100b+10c+d$.

Case I. Suppose $a = d$, and $b = c$. The probability for this is 1/100.

In this case, the difference between the original number and its reverse will be 0000; and the sum of this number and its reverse will also be 0000.

Case II. Suppose $a \neq d$, and $b = c$. The probability for this is 9/100.

In this case, the difference between the original number and its reverse will be $999(a-d)$ or $999(d-a)$, whichever is positive, the number in parentheses being some integer between 1 and 9, inclusive, which we will call e. Then the four digits making up the difference between the original number and its reverse will be, respectively, $(e-1)$, 9, 9, and $(10-e)$. Reversing this and adding will yield in every case 10989.

Case III. Suppose $a = d$, and $b \neq c$. The probability for this is 9/100.

In this case, the difference between the original number and its reverse will

be $90(b-c)$ or $90(c-b)$, whichever is positive, the number in parentheses being some integer between 1 and 9, inclusive, which we will call f. Then the four digits making up the difference between the original number and its reverse will be, respectively, 0, $(f-1)$, $(10-f)$, and 0. Reversing this and adding will yield in every case 0990.

Case IV. Suppose $a>d$, and $b>c$, or else $a<d$, and $b<c$. The probability for this is 81/200.

In this case, the difference between the original number and its reverse will be $999e+90f$. The four digits making up this number will be, respectively, e, $(f-1)$, $(9-f)$, and $(10-e)$. Reversing this and adding will yield 10890.

Case V. Suppose $a>d$, and $b<c$, or else $a<d$, and $b>c$. The probability for this is 81/200.

In this case, the difference between the original number and its reverse will be $999e-90f$. The four digits making up this number will be, respectively, $(e-1)$, $(10-f)$, $(f-1)$, and $(10-e)$. Reversing this and adding will yield 9999.

Summing up: 1% of all possible cases will yield 0000, 9% will yield 10989, 9% will yield 0990, 40½% will yield 10890, and 40½% will yield 9999.

If, in order to qualify as a four digit number, the first digit has to be other than zero, the solution will not be as neat. If, besides, initial zeros are dropped out before reversing the difference, the solution will become quite messy, and not interesting.

Also solved by I. A. Dodes, J. D. Haggard, S. W. Hahn, B. A. Hausmann, Vern Hoggatt and Octave Levenspiel (jointly), M. S. Klamkin, P. W. Allen Raine, Azriel Rosenfeld, C. A. Swanson, and the proposer.

Hahn generalized the problem to the situation where the four digit numbers are expressed in an arbitrary scale m. The proposer considered the extension to numbers of n digits.

The Circle Through the Feet of the Symmedians of a Triangle

E 1035 [1952, 551]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that if the circle passing through the feet of the symmedians of a non-isosceles triangle of sides a, b, c is tangent to one side, then the quantities b^2+c^2, c^2+a^2, a^2+b^2, arranged in some order, are consecutive terms of a geometric progression.

Solution by Chih-yi Wang, Hampton Institute. Let a, b, c be the sides BC, CA, AB of a non-isosceles triangle ABC, and A', B', C' the feet of the corresponding symmedians. For definiteness let the circle passing through A', B', C' be tangent to BC at A' and intersect CA, AB at M and N respectively. It is well known that

$$(1) \qquad BA'/A'C = c^2/b^2, \qquad CB'/B'A = a^2/c^2, \qquad AC'/C'B = b^2/a^2.$$

By the aid of (1) we obtain

$$A'C = ab^2/(b^2 + c^2), \qquad BA' = ac^2/(b^2 + c^2),$$

and analogous relations for CB', $B'A$, AC', and $C'B$, which together with the relations $(BN)(BC') = (BA')^2$, $AN = AB - NB$; $(CM)(CB') = (A'C)^2$, $AM = AC - MC$ imply

$$AN = c(b^4 + b^2c^2 + c^4 - c^2a^2)/(b^2 + c^2)^2$$

and

$$AM = b(b^4 + b^2c^2 + c^4 - a^2b^2)/(b^2 + c^2)^2.$$

By using the relation $(AB')(AM) = (AC')(AN)$, we get, after simplification,

$$(b^2 - c^2)(b^2 + c^2)^2 = (b^2 - c^2)(a^2 + b^2)(a^2 + c^2).$$

Since the triangle ABC is non-isosceles, the desired result follows.

Also solved by Arthur Gregory, Vern Hoggatt and Octave Levenspiel (jointly), L. M. Kelly, B. R. Leeds, Roscoe Woods, and the proposer.

Hoggatt showed that $(a^2+b^2)/(b^2+c^2)$, the ratio of the progression, is equal to $A'B'/A'C'$. Gregory's solution employed cartesian analysis.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4533. *Proposed by R. Kissling, Student, University of California, Berkeley*

Given $a^n+b^n=c^n$, with a, b, c, n integers and $a>b>1$, $n\geqq 2$; $\sigma_n(k)$ being the sum of the nth powers of all divisors of k, prove

$$\left|1 - \frac{\sigma_n(c)}{\sigma_n(a) + \sigma_n(b)}\right| < \frac{2n-1}{n(n-1)} < \frac{2}{n-1}, \tag{1}$$

$$\lim_{k\to\infty} \frac{\sigma_k(c)}{\left\{[\sigma_k(a)]^{n/k} + [\sigma_k(b)]^{n/k}\right\}^{k/n}} = 1. \tag{2}$$

4534. *Proposed by Frank Harary, University of Michigan*

If n is an odd prime and M is a symmetric n by n matrix each of whose rows is a permutation of $1, \cdots, n$, then the main diagonal of M is also such a permutation.

4535. *Proposed by M. S. Robertson, Rutgers University*

Let the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, be regular for $|z| \leqq 1$ and let it map $|z| \leqq 1$ onto a simply-connected domain D, star-like when viewed from the origin. Show that

$$2\left(\sum_{n=1}^{\infty} n\,|a_n|^2\right)^2 \leqq \sum_{n=1}^{\infty} (n+1)\,|a_1a_n + a_2a_{n-1} + \cdots + a_na_1|^2,$$

with equality only for $f(z) = a_1 z$.

4536. *Proposed by C. E. Stanaitis, St. Olaf College, Northfield, Minn.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \sin n\theta}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos n^2\theta \sin n\theta}{n}$$

are uniformly convergent in any interval, and that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \cos n\theta}{n}$$

is divergent.

4537. *Proposed by Albert Wilansky, Lehigh University*

Given that $\sum_n \sum_k a_{nk}x_k$ converges (as an iterated sum) whenever $\{x_k\}$ is a sequence such that $\sum |x_k| < \infty$. Show that $\sum_n \sum_k a_{nk}x_k = \sum_k \sum_n a_{nk}x_k$ for each such $\{x_k\}$.

SOLUTIONS

Three Related Triangles

4470 [1952, 46]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, three lines a', b', c' drawn through the vertices A, B, C determine by their intersections a triangle $A'B'C'$, and their isogonals a'', b'', c'' determine a triangle $A''B''C''$. (1) Show that the orthic triangles of $A'B'C'$ and $A''B''C''$ have equal perimeters. (2) If, further, a', b', c' are equally inclined to AB, BC, CA, show that the circles $A'B'C'$ and $A''B''C''$ are symmetric with respect to the line joining the symmedian point to the circumcenter.

Solution by the Proposer. (1) Let $\Delta, \Delta', \Delta''$ denote the areas of triangles ABC, $A'B'C'$, $A''B''C''$, and R, R', R'' their circumradii. Now, if a', b', c' divide angles A, B, C into angles α and α', β and β', γ and γ', one can show that

$$\frac{\Delta'}{\Delta} = \frac{(\sin\alpha \sin\beta \sin\gamma - \sin\alpha' \sin\beta' \sin\gamma')^2}{\sin A \sin B \sin C \sin A' \sin B' \sin C'} = \frac{4R^2R'^2D^2}{\Delta\Delta'},$$

where D^2 is the numerator of the middle fraction. Similarly,

$$\Delta''/\Delta = 4R^2R''^2D^2/\Delta\Delta''.$$

It now follows that $\Delta'/R' = \Delta''/R''$. But, if triangles $A'B'C'$ and $A''B''C''$ are acute, $2\Delta'/R'$ and $2\Delta''/R''$ are the perimeters of their orthic triangles.

If either of the triangles $A'B'C'$, $A''B''C''$ is not acute, the statement of the theorem needs modification. For example, if triangle $A'B'C'$ is obtuse angled at A', then the perimeter $2p'$ of its orthic triangle must be replaced by the quantity $2(p' - s_{A'})$, where $s_{A'}$ is the side of the orthic triangle corresponding to vertex A'.

(2) If $\alpha = \beta = \gamma$ then $A' = B''$, $B' = C''$, $C' = A''$, whence, from above, $\Delta'/\Delta = \Delta''/\Delta$, or $R' = R''$. One can also show, using barycentric coordinates, that the radical axis of the circles $A'B'C'$, $A''B''C''$ coincides with the Brocard diameter of triangle ABC. This completes part (2).

Also solved by Joseph Langr.

The Moduli of the Zeros of Polynomials

4474 [1952, 109]. *Proposed by Ky Fan, University of Notre Dame*

Let the zeros $z_i (1 \leqq i \leqq n)$ of the polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n$$

be so arranged that $|z_1| \geqq |z_2| \geqq \cdots \geqq |z_n|$. It is a classical result due to Carmichael and Mason that

$$|z_1| \leqq \sqrt{1 + \sum_{i=0}^{n-1} |a_i|^2} = B. \tag{1}$$

Using Jensen's inequality, M. Fujiwara obtained a stronger inequality

$$|z_1 z_2 \cdots z_k| \leqq B, \qquad (1 \leqq k \leqq n). \tag{2}$$

Prove that this result (2) can still be slightly improved as follows:

$$|z_1 z_2 \cdots z_k|^2 \leqq \tfrac{1}{2}(B^2 + \sqrt{B^4 - 4|a_0|^2}), \qquad (1 \leqq k \leqq n). \tag{3}$$

Solution by the Proposer. Using the coefficients of the polynomial $f(z)$, we form the matrix of order n:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 & 0 \\ . & . & . & . \cdots & . & . \\ 0 & 0 & 0 & 0 \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

Then the characteristic equation of this matrix A is $f(z) = 0$. In other words, the zeros of our polynomial $f(z)$ are exactly the eigenvalues of A. Let A^* denote the complex conjugate transpose of A, i.e., the (i, j)-element of A^* is the complex conjugate of the (j, i)-element of A. Form the product AA^*:

$$AA^* = \begin{pmatrix} 1 & 0 & 0 \cdots & 0 & -\bar{a}_1 \\ 0 & 1 & 0 \cdots & 0 & -\bar{a}_2 \\ 0 & 0 & 1 \cdots & 0 & -\bar{a}_3 \\ . & . & . \cdots & . & . \\ 0 & 0 & 0 \cdots & 1 & -\bar{a}_{n-1} \\ -a_1 & -a_2 & -a_3 \cdots & -a_{n-1} & \sum_{i=0}^{n-1} |a_i|^2 \end{pmatrix}.$$

Then a simple calculation gives the characteristic equation of the matrix AA^*:

$$(w-1)^{n-2}\left\{w^2 - \left(1 + \sum_{i=0}^{n-1} |a_i|^2\right)w + |a_0|^2\right\} = 0.$$

In this we have assumed $n \geqq 2$, the case $n = 1$ of our problem being trivial.

Hence the eigenvalues of AA^* are:

$$w_1 = \frac{1}{2}\left\{1 + \sum_{i=0}^{n-1} |a_i|^2 + \sqrt{\left(1 + \sum_{i=0}^{n-1} |a_i|^2\right)^2 - 4|a_0|^2}\right\},$$

$$w_2 = w_3 = \cdots = w_{n-1} = 1,$$

$$w_n = \frac{1}{2}\left\{1 + \sum_{i=0}^{n-1} |a_i|^2 - \sqrt{\left(1 + \sum_{i=0}^{n-1} |a_i|^2\right)^2 - 4|a_0|^2}\right\}.$$

It can be easily seen that $w_1 \geqq 1 \geqq w_n$.

Now, our inequality (3) is an immediate consequence of the following result of H. Weyl, Inequalities between two kinds of eigenvalues of a linear transformation, *Proc. National Acad. Sci. U.S.A.*, v. 35, 1949, pp. 408–411: *For an arbitrary $n \times n$ matrix A, if the eigenvalues z_i of A and the eigenvalues w_i of AA^* are so*

arranged that $|z_i| \geqq |z_{i+1}|$, $w_i \geqq w_{i+1}$, *then*

$$|z_1 z_2 \cdots z_k|^2 \leqq w_1 w_2 \cdots w_k, \qquad (1 \leqq k \leqq n).$$

It is hoped that a direct, matrix-free proof of (3) may yet be submitted.

Generalization of the Fundamental Theorem of Algebra

4475 [1952, 110]. *Proposed by D. J. Newman, Harvard University*

Let $f(z)$ be continuous throughout the complex z-plane, and suppose that $f(z)/z \to 1$ as $z \to \infty$. Show that $f(z)$ must have a zero.

I. *Solution by Ramon Moore, Ballistic Research Laboratory, Aberdeen, Md.* From the hypothesis there is a real number $R>0$, sufficiently large, such that

$$\left|\frac{f(z)}{z} - 1\right| < 1 \quad \text{or} \quad |f(z) - z| < |z|$$

for $|z| > R$. Hence as z traces out a circle C with center at the origin of the complex plane and radius $t_0 > R$, $f(z)$ will trace out a closed curve Γ in whose interior the origin lies. But $f(0)$ is some point ζ in the complex plane and since $f(z)$ is a continuous function of z, as the radius t of circles traced out by z various continuously from t_0 to 0 the closed curve traced out by $f(z)$ will vary continuously from Γ to the single point ζ. Hence there is some $\bar{t}$ for which the curve traced out by $f(z)$, as z traces out the circle of radius $\bar{t}$, passes through the origin of the complex plane.

II. *Solution by J. W. Gaddum, National Bureau of Standards, Los Angeles.* Let $g(z) \equiv z - f(z)$. Then $g(z)$ is a continuous function mapping the whole z-plane into itself. Also, $g(z)/z$ approaches zero as $z \to \infty$. Let C_r be the closed region bounded by the circle of radius r and with center at the origin. Now for some r, g maps C_r into itself. Otherwise there would be an infinite sequence $z_1, z_2, \cdots$, with $|g(z_n)|/|z_n| > 1$, and $g(z_n)/z_n$ would not have the limit zero. For a C_r which is mapped by g into itself there is a fixed point, that is, a z_0 such that $g(z_0) = z_0$. Thus $f(z_0) = 0$.

Also solved by L. M. Kelly, J. H. Michael, E. J. Mickle, L. A. Ringenberg, and the Proposer.

Editorial Note. It is evident that the argument in I above holds also if the hypothesis is changed by replacing $f(z)/z$ by $f(z)/z^n$. $f(z)$ a polynomial is a special case.

Number of Unrestricted Partitions of n

4476 [1952, 110]. *Proposed by T. M. Apostol, California Institute of Technology*

Let $p(n)$ be the number of unrestricted partitions of n, and let $\sigma(n)$ be the sum of the divisors of n. Prove

$$p(n) = \sum_{m=1}^{n} \sum_{\substack{n_1, n_2, \cdots, n_m \\ n_1 + \cdots + n_m = n}} \frac{1}{m!} \frac{\sigma(n_1)\sigma(n_2) \cdots \sigma(n_m)}{n_1 \cdot n_2 \cdots n_m}.$$

For each fixed m, the inner sum is to be taken over all positive integers $n_1, n_2, \cdots, n_m$ whose sum is n, the order of the summands being taken into consideration.

Solution by Leonard Carlitz, Duke University. We have

$$\prod_{m=1}^{\infty} (1 - x^m) = 1 + \sum_{n=1}^{\infty} p(n) x^n.$$

Also

$$-\log \prod_{m=1}^{\infty} (1 - x^m) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^{mk}}{k} = \sum_{m=1}^{\infty} \frac{\sigma(n)}{n} x^n,$$

whence

$$1 + \sum_{n=1}^{\infty} p(n) x^n = \exp \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n$$

$$= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left[\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n \right]^m.$$

Since

$$\left[\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} x^n \right]^m = \sum_{n_1, \cdots, n_m = 1}^{\infty} \frac{\sigma(n_1) \cdots \sigma(n_m)}{n_1 \cdots n_m} x^{n_1 + \cdots + n_m},$$

it follows that

$$p(n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{n_1 + \cdots + n_m = n} \frac{\sigma(n_1) \cdots \sigma(n_m)}{n_1 \cdots n_m}.$$

In the same way, if we put

$$\prod_{m=1}^{\infty} (1 + x^m) = 1 + \sum_{n=1}^{\infty} q(n) x^n,$$

then

$$q(n) = \sum_{m=1}^{n} \frac{1}{m!} \sum_{n_1 + \cdots + n_m = n} \sigma'(n_1) \cdots \sigma'(n_m),$$

where

$$\sigma'(n) = \sum_{k \mid n} \frac{(-1)^{k-1}}{k}.$$

Also solved by the proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

The Nature of Number. An Approach to Basic Ideas of Modern Mathematics. By Roy Dubisch. New York, The Ronald Press Co. 1952. xii+159 pages. $4.00.

"The objectives of this book may be stated as follows: (1) To portray the development of a single line of mathematical thought from its most primitive beginnings of contemporary times, and (2) to make the abstractions of advanced mathematics easier to grasp by pointing out that the concreteness of elementary mathematics is largely illusory and that, actually, all basic mathematics is abstract. The particular line of procedure employed is that of the development of the number concept." With these objectives in mind, the author has written a book that should be stimulating reading for young students and will present some modern points of view to older people who have not followed recent developments. The book would also be an excellent foundation for a reading course—a type of instruction which, with the shortage of teachers, might become more popular in colleges. A few minor objections: The definition of transcendental number (p. 77) should be corrected and clarified. On p. 73, the author might well have proved that the square of an odd number is odd instead of saying merely that "the fact can be rigorously proved." A word about the scope of symbols would have been in line with the author's emphasis on notational questions. For instance, he might have explained that $2+3\cdot 4$ stands for $2+(3\cdot 4)$ rather than for $(2+3)\cdot 4$. On p. 34 the author quotes an (anonymous) reader of the manuscript as saying that geometry actually is a mixture of postulates, geometric intuition, and abracadabra. This statement might better have remained unquoted since it must have been made in a moment of absentmindedness when that reader forgot about the work of Pasch, Pieri, Hilbert, and others. But these are minor points. All in all, one would wish that more books of this type were written and that they were widely read.

KARL MENGER
Illinois Institute of Technology

Society of Actuaries' Textbook on Life Contingencies. By C. W. Jordan. Chicago, Society of Actuaries, 1952. xi+331 pages. $8.00.

"Life Contingencies" is the name applied to a branch of mathematics which deals with probabilities of human survivorship and which provides the basis for financial calculations involving a combination of these probabilities with functions of compound interest. Through the latter, the technical basis for a life

insurance policy is established; the life insurance applications of Life Contingencies theory are, in fact, its most important applications and the subject has developed for the most part along lines dictated by the needs of the life insurance companies.

Until now, the standard reference work in this field has been the textbook by E. F. Spurgeon, published by the British Institute of Actuaries. For many years there has been a need for a treatment of the subject from the American point of view, similarly comprehensive in scope but reflecting life insurance practices in this country rather than the practices which obtain in Great Britain. The Society of Actuaries therefore commissioned Mr. C. W. Jordan, a Fellow of the Society and Associate Professor of Mathematics at Williams College, to write the present volume.

Mr. Jordan has combined broad knowledge of his subject with great pedagogic ability derived from his experience as a teacher, and has written a book of which the Society of Actuaries is understandably proud and which students will respect. He has stated clearly the fundamental principles upon which the theory is built and has developed these principles in a way which emphasizes throughout the essential unity of the different parts of the theory.

Actuaries naturally comprise the group most interested in this branch of mathematics, but the subject should also be of interest to others both on its merits as a practical application of probability theory and because the methods employed may find application in other fields.

D. H. HARRIS
Equitable Life Assurance
Society of the United States

Theory of Numbers. By B. M. Stewart, New York, The Macmillan Company, xiii+267 pages, $5.50.

The preface of this book opens with the words "The present work makes no pretense at being more than a textbook; the subject matter is classical and the only attempt at originality is in the choice of topics and the manner of presentation." In this attempt the author has been highly successful.

The subject matter is motivated in detail and the explanations are full. There is none of that forbidding conciseness which often makes a textbook unpleasant to students. A simple illustration is the statement *verbatim* of Theorem G, 8.1. "If $(a, m)=d$, then $ax\equiv b \bmod m$ has no solutions when d is not a divisor of b; but if d divides b, there are exactly d solutions." There are exercises and illustrations to satisfy all tastes; they vary from games to group-theoretical questions. Unlike other American writers the author does not hesitate to introduce simple concepts from group and set theory, and to use the advantages they have to offer.

In addition to the standard topics (divisibility, congruences, quadratic residues, forms, additive number theory) there is an introduction to algebraic

numbers *via* the question of unique factorization, and a full development of the real number system from Peano's axioms. One can take exception to a strict interpretation of the author's assertion (p. 205) that these axioms are categorical. On the other hand he avoids the classical blunder of an incomplete definition of addition from the axioms (p. 201).

One can also object to the assertion (p. 205) that "Ordinarily it is rather restricted and uninteresting to study a mathematical system that is categorical." Finite dimensional vector spaces and separable Hilbert space seem to me adequate counterexamples. But this is a small matter compared to the author's achievement.

HARRY POLLARD
Cornell University and Institute for Advanced Study

NEW BOOKS RECEIVED

A Short Table for the Bessel Functions $I_{n+1/2(x)}$, $(2/\pi)K_{n+1/2(x)}$. By C. W. Jones. Published for the Royal Society at the University Press, Cambridge, 1952. 20 pages. 6s. 6d.

Advanced Mathematics in Physics and Engineering. By Arthur Bronwell. New York, McGraw-Hill Book Company, 1953. xvi+475 pages. $6.00.

Anschauliche Einführung in die Mehrdimensionale Geometrie. By Walther Lietzmann. Germany, R. Oldenbourg, Publisher, Munich. 1952.

Computing Manual. By Fred Gruenberger. Madison, Wisconsin. The University of Wisconsin Press, 1952. 123 pages. $3.00.

Mathematics of Finance. By L. L. Smail. New York, McGraw-Hill Book Company, 1953. x+282 pages. $4.50.

Calculus, A Modern Approach. By Karl Menger. Printed by Illinois Institute of Technology (Mimeographed form). 1952. xxv+255 pages. $4.50.

Associated Measurements. By M. H. Quenouille. New York, Academic Press, Inc., 1952. x+242 pages. $5.80.

CLUBS AND ALLIED ACTIVITIES

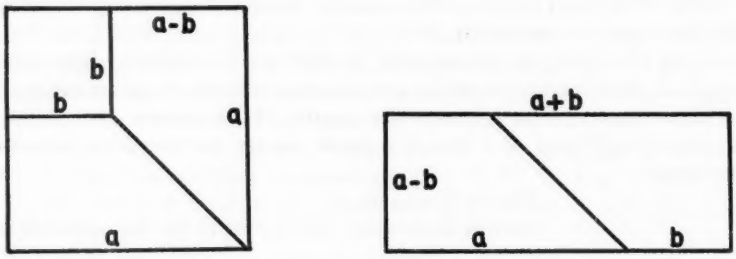
EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

(a^2-b^2): A RECREATION

The familiar formula for factoring a^2-b^2 can be made the basis of an elementary recreation, perhaps suitable for secondary school students for whom the available recreational material in algebra is not too plentiful.

1. A geometric proof of the formula, $a^2-b^2=(a+b)(a-b)$.
Behold!



2. A proof that a rectangle of given perimeter has the greatest area when its sides are equal.

Let the perimeter of the rectangle be $4a$, with a constant, and let the sides be $a+b$ and $a-b$. Then the area of the rectangle is $(a+b)(a-b)=a^2-b^2$. Clearly, this is a maximum when $b=0$.

3. A short method of squaring numbers near 50, 100, 500, 1000, *etc.*

We use the rule, $a^2=(a+b)(a-b)+b^2$. This rule is very useful if one has memorized the squares of the first 25 integers. For example,

$$83^2 = (83 + 17)(83 - 17) + 17^2 = (100)(66) + 289 = 6889.$$

$$978^2 = (978 + 22)(978 - 22) + 22^2 = (1000)(956) + 484 = 956,484.$$

4. A method of finding the product of certain small numbers.

If, as in (3), one has memorized the squares of the first 25 integers (or more), he can reduce many multiplications to the difference of two squares. For example,

$$19 \cdot 13 = (16 + 3)(16 - 3) = 256 - 9 = 247,$$
$$26 \cdot 14 = (20 + 6)(20 - 6) = 400 - 36 = 364.$$

5. Interesting number forms.

$$6^2 - 5^2 = 11$$
$$56^2 - 45^2 = 1111$$
$$556^2 - 445^2 = 111111$$
$$5556^2 - 4445^2 = 11111111$$
$$55556^2 - 44445^2 = 1111111111$$

etc.

These and other forms are derived easily. Thus, let us choose $a+b=1001$ and $a-b=333$. Then $a=667$, $b=334$, and $667^2-334^2=333333$. Indeed,

$$7^2 - 4^2 = 33$$
$$67^2 - 34^2 = 3333$$
$$667^2 - 334^2 = 333333$$
$$6667^2 - 3334^2 = 33333333$$
$$66667^2 - 33334^2 = 3333333333$$
$$\textit{etc.}$$

6. A method for finding square roots.

We use the rule, $a^2=b^2+k(a+b)$, where $k=a-b$. For example, to find $\sqrt{330}$:

$$\begin{aligned}
18^2 &= 324 \\
.2(18 + 18.2) &= \underline{\quad 7.24} \qquad (k = .2) \\
18.2^2 &= 331.24 \\
-.04(18.2 + 18.16) &= \underline{-1.4544} \qquad (k = -.04) \\
18.16^2 &= 329.7856 \\
.005(18.16 + 18.165) &= \underline{\quad .181625} \qquad (k = .005) \\
18.165^2 &= 329.967225 \\
.0009(18.165 + 18.1659) &= \underline{\quad .03269781} \qquad (k = .0009) \\
18.1659^2 &= 329.99992281
\end{aligned}$$

Can any reader add to this list?

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

INTERNATIONAL PRIZE IN HONOR OF GUIDO FUBINI

In memory of Guido Fubini, l'Unione Matematica Italiana will award an international prize in mathematics which has been donated by his friends and admirers.

There will be a single award made in Italian currency equivalent to about 550 grams of gold, and it will be for important contributions in the field of dif-

ferential geometry published between January 1, 1946 and December 31, 1953.

Applicants are invited to submit lists of such works to the Presidency of the Unione Matematica Italiana before March 31, 1954, but the awarding committee, whose decision will be final, may also consider works published within this period by mathematicians who do not so apply.

If there are no suitable works in differential geometry, the committee may also grant the prize for contributions made in the same period to the theory of automorphic functions and related topics.

The committee will consist of: Professor Salomon Bochner, Princeton University; Professor Charles Ehresmann, University of Strasbourg, France; Professor Alessandro Terracini, University of Turin, Italy.

MATHEMATICS INSTITUTE IN RIO DE JANEIRO

An Institute for Pure and Applied Mathematics has been founded in Rio de Janeiro by the National Research Council of Brazil. The new institution will sponsor the publication of the principal Brazilian mathematical journal *Summa Brasiliensis Mathematicae*. The Director of the Institute is the noted mathematician and astronomer, Dr. Lelio I. Gama, Caixa Postal 46, Rio de Janeiro, Brazil.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1953:

Boston College. June 24 to August 4: Professor Calabi, group theory; Professor Eiardi, advanced calculus and differential equations.

The Catholic University of America. June 29 to August 8: Professor Ramler, college geometry, analytic projective geometry, differential equations; Professor Finan, fundamental concepts; Professor Rice, advanced calculus II; Professor Moller, higher algebra I; Miss Handrich, advanced calculus I.

Columbia University, Teachers College. July 6 to August 14: Professor Fehr, teaching arithmetic in the elementary school; Professors Fehr and Rosskopf, research and departmental seminar in teaching mathematics; Professors Fehr and Yates, professionalized subject matter in advanced secondary school mathematics, part I; Professor Rosskopf, foundations of mathematics for teachers; Professors Rosskopf and Shuster, teaching geometry in secondary schools; Professors Rosskopf and Yates, teaching of elementary college mathematics; Miss Schult, supervision and teaching of mathematics in the junior high school; Professor Shuster, field work in mathematics, business arithmetic and mathematics. July 20 to 31: Professors Fehr, Yates and special lecturers, workshop in the theory, construction and use of models and materials in mathematical education; daily from 1:30 to 4 p.m.

De Paul University. June 15 to August 4: Professor De Cicco, tensor analysis and Riemannian geometry, matrices and linear transformations; Professor Caton, theory of oscillations, college geometry.

Kent State University. June 22 to July 31: Professor Jenkins, theory of numbers; Professor Dressler, theory of equations; Professor Stapleford, advanced methods of teaching mathematics in high school. August 3 to September 4: Professor Iwanchuk, vector analysis.

Massachusetts Institute of Technology. July 6 to 10: Professors Norbert Wiener and Y. W. Lee will conduct a one week special program entitled "Mathematical Problems of Communication Theory."

Michigan State College. June 23 to July 31: Professor Bell, matrices and groups, higher algebra II; Mr. Coy, elements of statistics, correlation analysis; Professor Herzog, complex variable III; Dr. J. Kelly, theory of equations, differential equations; Professor L. Kelly, topology, foundations of mathematics; Professor Powell, complex variable I, advanced calculus I. June 23 to August 21: Professor Arnold, analysis of variance, theory of probability; Professor Hill, real variable, advanced calculus II; Professor Olkin, advanced topics in statistics; Professor Parkus, potential theory, partial differential equations; Professor Stelson, solid analytic geometry.

Northwestern University. June 19 to August 22: differential equations; theory of statistics; introduction to the theory of numbers; algebra for teachers; introduction to the theory of groups; vector analysis; analysis seminar.

Purdue University. June 15 to August 8: partial differential equations and applications; advanced calculus I; vector analysis; introduction to theoretical statistics; higher algebra; fundamental concepts of mathematics; functions of a complex variable; Laplace and Fourier transformations; theory of groups; statistical methods; machine methods in statistics; advanced statistical methods. June 22 to August 1: special program for the General Electric Mathematics Fellows; fifty fellowships for secondary school teachers have been established at the University by the General Electric Company.

Syracuse University. June 29 to August 7: Professor Kibbey, introduction to mathematical logic; Professor Exner, analysis of elementary mathematics; Mr. Paul Smith, teaching of high school mathematics, workshop in mathematics education (these courses are specially adapted for secondary school teachers).

University of Buffalo. June 29 to August 8: Professor Schneckenburger, topics in higher geometry; Professor Gehman, foundations of mathematics; Professor Montague, advanced methods for teachers of mathematics.

University of California, Berkeley. June 22 to August 1: Professor Kendall, stochastic processes associated with population growth and with the theory of queues; Professor Neyman, individual research leading to higher degrees. August 3 to September 12: Professor Neyman, individual research leading to higher degrees.

University of Chicago. June 22 to August 29: Professor Halmos, algebra IV (theory of groups, commutative rings), topics in set theory; Professor Hartung, fundamental concepts of mathematics for teachers; Professor Kaplansky, differential geometry; Professor Segal, point set topology, representation of topo-

logical and Lie groups; Professor Stone, Hilbert space; Professor Weil, elementary number theory, seminar on current literature; Professor Zygmund, ordinary differential equations.

University of Colorado. June 15 to August 25: Professor Chowla, advanced calculus, introduction to modern algebra; Mr. Osserman, topology; Professor Britton, vector analysis, functions of a complex variable. June 15 to July 21: Professor Jones, fundamental concepts of arithmetic and algebra; Professor Pingry (University of Illinois), teaching of secondary mathematics, mathematics workshop in curriculum problems. July 22 to August 25: Professor Kendall, history of mathematics.

University of Florida. June 15 to August 15: Professor Cowan, Fourier series; Professor Dostal, vector analysis; Professor Ellis, theory of groups of finite order, differential geometry; Professor Gager, history of elementary mathematics; Professor Gormsen, synthetic projective geometry, advanced college geometry; Professor Lang, functions of a complex variable II; Professor Phipps, foundations of geometry; Professor Smith, advanced topics in calculus I and II; Professor South, theory of probability and theory of sampling.

University of Kentucky. June 22 to August 15: Professor Cowling, advanced calculus; Professor Pence and Miss Baskett, college geometry; Professor Leser, introduction to applied mathematics; Professor Goodman, introduction to the theory of numbers; Professor Ward, higher algebra.

University of Maryland, Department of Mathematics. June 22 to July 31: Professor Good, theory of equations; Professor Jackson higher geometry; Mr. Spencer, vector analysis; Dr. Payne, selected topics in applied mathematics.

University of Michigan. June 22 to August 14: Professor Bartels, hydrodynamics; Professor Bott, operational mathematics; Professor Carver, theory of statistics II and finite differences; Professor Christofferson, teaching of geometry; Professor Copeland, probability and theory of games; Professor Craig, theory of statistics I and multivariate analysis; Professor Dwyer, computational methods; Professor Jones, history of algebra; Professor Leisenring, higher geometry for teachers; Professor Nevanlinna, advanced functions of complex variables; Professor LeVeque, theory of numbers; Professor Lohwater, elementary functions of a complex variable with applications; Professor Myers, functions of a real variable and calculus of variations; Professor Nesbitt, mathematics of life insurance; Professor Nyswander, Galois theory and theory of matrices; Professor Rainville, intermediate differential equations; Professor Rothe, Fourier series and methods in partial differential equations; Professor Tornheim, arithmetic of rings; Professor Young, foundations of mathematics and unified topology.

University of North Carolina. June 11 to July 17: Professor Winsor, elementary algebra from an advanced viewpoint; Professor Garner, history of mathematics; Professor Cameron, fundamental concepts; Professor Linker, differential equations; Professor Mann, topics in analysis; Professor Brauer, some recent results in number theory. July 20 to August 22: Professor Mackie, theory of

equations; Professor Lasley, synthetic projective geometry; Professor Jones, topics in analysis; Professor MacNerney, summability.

University of Pittsburgh. June 8 to July 17: Professor Bryson, differential equations; Professor Blumberg, advanced calculus; Professor Taylor, functions of a complex variable; Professor Laush, functions of a real variable; Professor Elyash, advanced function theory; Professor Bompiani, advanced differential equations. July 20 to August 28: Professor Bryson, differential equations; Professor Blumberg, advanced calculus; Professor Taylor, functions of a complex variable; Professor Laush, functions of a real variable, Lie theory of groups; Professor Bompiani, advanced differential equations (all courses in the July 20 to August 28 session except Lie theory of groups are continuations of courses of the preceding six weeks). June 15 to August 7 (evenings): Professor Barsotti, partial differential equations and Fourier series. June 29 to August 7: Professor Teats, history of mathematics; Professor Blumberg, recreational mathematics for teachers, theory of equations; Mr. Kachun, navigation for teachers.

University of Southern California. June 22 to August 1: Professor Steed, secondary mathematics from an advanced standpoint; Professor Sherman, foundations of mathematics; Professor Whaples (Indiana University), advanced theory of numbers, seminar in algebra; (evenings) advanced calculus; introduction to theory of complex variables; seminar.

University of Washington. June 22 to July 23 and July 24 to August 21: Professor Peterson, linear algebra; Professors Ball and Cramlet, differential equations; Professors McFarlan and Avann, vector analysis; Professor Hewitt, Fourier series; Professor Dekker, advanced Euclidean geometry.

PERSONAL ITEMS

Professor L. S. Hill of Hunter College represented the Association at the inauguration of President B. G. Gallagher of the City College of the City of New York on February 19, 1953.

Professor A. A. Albert of the University of Chicago has been elected to a corresponding membership in the Academia Brasileira de Ciências.

Professor H. B. Curry of Pennsylvania State College has been granted the title of "Visiting Professor Honoraire" of the University of Louvain, Belgium.

Dr. Leila A. Dragonette, research associate at the University of Chicago, is the recipient of the 1952 Research Award of $500 of Sigma Delta Epsilon. This award was made for a paper entitled "Some Asymptotic Formulae for the Mock Theta Series of Ramanujan."

Professor W. L. Duren, Jr. of Tulane University has been appointed to a Faculty Fellowship by the Ford Foundation and is on leave of absence from the University for the year 1952–53.

Associate Professor Maurice L'Abbé of the University of Montreal has been awarded a post-doctoral fellowship by the Government of Canada and is spending the year in France.

Professor C. O. Oakley has received a Ford Foundation grant and is on leave

from Haverford College for the year 1952–53.

Dr. E. R. Reifenberg, Fellow of Trinity College, Cambridge, holds a Commonwealth Fund Fellowship for 1952–53 and is spending the year at the University of California, Berkeley.

Professor Moses Richardson of Brooklyn College has a Ford Foundation Fellowship for the year 1952–53 and is at Princeton University.

Dr. I. M. Rose of the University of Massachusetts has been granted a Ford Foundation Fellowship and is on leave of absence for the current academic year.

Assistant Professor E. H. Spanier and Professor André Weil of the University of Chicago have received Guggenheim Fellowships for 1952–53.

Boston University announces the following appointments to instructorships: Miss Elizabeth A. Shuhany and Mr. V. R. Staknis.

Brooklyn College reports the following: Professor R. A. Johnson, formerly chairman of the Department of Mathematics, has retired; Associate Professor Samuel Borofsky has been appointed Chairman of the Department; Assistant Professor Moses Richardson has been promoted to an associate professorship; Dr. Melvin Hausner, formerly a mathematician with the Rand Corporation, has been appointed to an instructorship; Mrs. Mary D. Boeker has retired.

Carleton College announces the following: Associate Professor K. O. May has been promoted to a professorship; Dr. Anne W. Calloway and Dr. J. M. Calloway have been appointed to assistant professorships; Mr. F. L. Wolf has been appointed to an instructorship.

At Colorado Agricultural and Mechanical College: Professor A. G. Clark, formerly head of the Department of Mathematics, now has the position of Dean of the College and Professor of Mathematics; Professor H. T. Guard has been promoted to the position of Head of the Department.

Creighton University announces the promotions of Associate Professor A. K. Bettinger to a professorship and of Instructor J. P. Markoe to an assistant professorship.

Hampton Institute reports the following: Professor J. D. Eshleman has retired with the title of Professor Emeritus; Miss Rosalind M. Eagleson has been promoted to an assistant professorship.

Iowa State College makes the following announcements: Dr. H. E. Goheen, who has been Assistant Professor of Electrical Engineering at the University of Pennsylvania, has been appointed to an associate professorship; Mr. G. W. Peglar and Mr. David Wend have been appointed to instructorships; Dr. A. M. Feyerherm has been promoted to an assistant professorship; Associate Professor Gertrude A. Herr has retired; Assistant Professor Carl Langenhop has returned after a year's leave of absence at Princeton University; Associate Professor C. G. Maple has returned after a two years' leave of absence with the United States Navy in Washington.

Massachusetts Institute of Technology announces the following appointments: Dr. Kenkichi Iwasawa, formerly a member of the Institute for Advanced Study while on leave from his position as Assistant Professor at the University

of Toronto, has been appointed to an assistant professorship; Dr. Joseph Sampson of Princeton University and University of Paris, Dr. J. B. Serrin, Jr., Princeton University, and Dr. Gerard Washnitzer, also of Princeton University, have been appointed to C. L. E. Moore instructorships.

Northwestern University announces: Dr. Meyer Dwass of the United States Census Bureau has been appointed to an assistant professorship; Dr. Daniel Resch of the University of Michigan and Dr. Maxwell Rosenlicht of Princeton University have been appointed to assistant professorships; Dr. Jacob Dekker of the University of Chicago and Dr. Alex Rosenberg, Air Force Project, University of Michigan have been appointed to instructorships; Dr. M. P. Gaffney is on leave of absence until September 1, 1953 and is at Princeton University where he is working on a research project for the Army; Professor E. J. Moulton has retired with the title of Professor Emeritus.

Pennsylvania State College reports the following: Dr. R. D. Ayoub of Harvard University has been appointed to an assistant professorship; Assistant Professor Aline H. Frink has been promoted to an associate professorship; Instructors J. H. Kinney and William Craig have been promoted to assistant professorships.

Purdue University announces the following: Associate Professor H. F. S. Jonah has been promoted to a professorship; Visiting Professor Lamberto Cesari has been appointed to a professorship; Dr. J. H. B. Kamperman of Amsterdam, Holland has been reappointed Visiting Professor; Dr. D. B. Owen and Dr. A. J. Perlis have been appointed to assistant professorships; Mr. Leonard Gillman, formerly of the Navy Evaluations Group, has been appointed to an instructorship; Associate Professor R. B. Stone, formerly registrar, has retired with the title of Professor Emeritus.

Stevens Institute of Technology announces the following appointments to instructorships: Mr. Emanuel Fischer, formerly research assistant at Institute of Mathematics and Mechanics, New York University; Mr. Henry Polowy, previously instructor at Nutley High School, New Jersey.

Trinity College, Hartford, Connecticut announces the establishment of a new part-time graduate program leading to the degree of Master of Science in Mathematics.

Tulane University announces the following appointments: Dr. Morris Friedman of the University of Chicago has been appointed Visiting Assistant Professor; Mr. R. E. Allan and Dr. A. L. Shields, formerly graduate fellow at Massachusetts Institute of Technology, have been appointed to instructorships.

At the United States Naval Academy: Assistant Professor J. W. Popow has been promoted to an associate professorship; Senior Professor J. N. Galloway and Professor John Tyler have retired.

University of Alberta reports the following: Assistant Professor E. Phibbs has been promoted to an associate professorship; Lecturer T. M. Fostvedt, Dr. G. K. Horton, and Lecturer R. C. Jacka have been promoted to assistant professorships.

University of British Columbia announces: Dr. G. E. Latta of Princeton University has been appointed to an assistant professorship; Professors R. D. James and S. A. Jennings were holders of summer fellowships of the Canadian Mathematical Congress at Queen's University, Kingston, Ontario, during the summer of 1952.

At University of California at Berkeley: Mr. Kurt Bing has been appointed Associate; Professor Salomon Bochner of Princeton University is Visiting Professor during the second semester of 1952–53; Miss Anne C. Davis has been appointed to the position of junior research mathematician on a research project of the Office of Ordnance; Dr. Ulf Grenander was Visiting Associate Professor during the first semester of 1952–53; Dr. Lucien LeCam and Dr. Maurice Sion have been appointed to instructorships; Dr. R. F. Tate has been appointed Lecturer; Associate Professor Frantisek Wolf has been promoted to a professorship; Assistant Professors E. W. Barankin and Edmund Pinney have been promoted to associate professorships; Dr. H. M. Hughes and Dr. R. M. Lakness have been promoted to assistant professorships; Associate Professor E. W. Barankin is on leave of absence during 1952–53 and is at the Institute for Numerical Analysis; Professor D. H. Lehmer is also on leave during 1952–53 and is serving as Head of the Institute for Numerical Analysis; Professor Edmund Pinney was on sabbatical leave during the fall of 1952; Professor Sophia L. McDonald was on sabbatical leave in residence during the fall of 1952; Professor G. C. Evans is on sabbatical leave in residence, spring, 1953; Professor Thomas Buck has retired.

University of Chicago announces that Assistant Professors Irving Kaplansky and I. E. Segal have been promoted to associate professorships.

University of Maryland reports: Dr. Stuart Haywood, formerly supervisor of mathematics in the University of Maryland program in Europe, and Dr. D. M. Young, previously at Aberdeen Proving Ground, have been appointed to assistant professorships; Dr. H. S. Collins, previously a graduate student at Tulane University, Mrs. Elizabeth Cuthill of Purdue University, Dr. Charles McArthur, formerly a graduate assistant at Tulane University, Miss Jacqueline Penez, formerly research assistant at the University of Minnesota, Dr. W. R. Thickstun, who has been a graduate assistant at the Institute of Fluid Dynamics of the University, have been appointed to instructorships; Mr. B. R. Cato of the University of Arizona has been appointed to a junior instructorship.

At the University of Texas: Dr. W. T. Guy has been promoted to an assistant professorship; Dr. F. N. Edmonds, Jr., previously assistant professor of astronomy at the University of Missouri, has been appointed to an assistant professorship in applied mathematics and astronomy; Associate Professor R. E. Greenwood has returned after two years' military leave for service as Lieutenant Commander in the United States Naval Reserve.

University of Toronto reports: Dr. G. F. D. Duff of Massachusetts Institute of Technology has been appointed to an assistant professorship; Dr. G. A. Dirac of King's College, London, has been appointed Lecturer; Lecturers A. J. Cole-

man, R. G. Stanton, and W. T. Tutte have been promoted to assistant professorships; Dean S. Beatty who has served also as Head of the Department of Mathematics has retired with the title of Dean Emeritus after forty-four years at the University; I. Guttman, B. J. Kirby, K. Okashimo, G. W. Schaefer, S. Schuster, D. A. Sprott, J. R. Walter, and M. S. Watkins have been appointed to teaching fellowships.

University of Western Ontario announces the following: Lecturer G. P. Henderson has been promoted to an assistant professorship; Professor H. R. Kingston, Principal of University College and Dean of Arts and Science, has retired.

Washington University announces the appointments of Mr. W. A. Couch and Mr. G. C. Cree to instructorships.

Wheaton College, Illinois, makes the following announcements: Associate Professor Fannie W. Boyce has been promoted to a professorship; Assistant Professor Angeline J. Brandt has been promoted to an associate professorship.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 125 persons have been elected to membership by the Board of Governors on applications duly certified.

D. W. ALLAN, Student, University of Toronto.

S. I. ALLEN, M. A. (Harvard) Instr., University of Massachusetts.

NACHMAN ARONSZAJN, D.Sc. (Paris) Professor, University of Kansas.

R. T. BARNES, Student, University of Buffalo.

J. S. BECKER, B.S. (Illinois) Conductor, New York Central Railroad, South Bend, Ind.

W. A. BEYER, M.S. (Illinois) Instr., Pennsylvania State College.

R. L. BLAIR, Ph.D. (Iowa) National Science Foundation Fellow, University of Chicago.

R. D. BOSWELL, JR., M.S. (Mississippi S. C.) Professor, Reinhardt College, Waleska, Ga.

A. R. BROWN, JR., Ph.D. (Yale) Mathematician, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

G. C. BUSH, Student, McMaster University.

D. I. CAPLAN, M.S. (M.I.T.) Engineering Assistant, Burroughs Adding Machine Company, Philadelphia, Pa.

REV. R. E. CARIAS, B.A. (Javeriana U., Bogota) Grad. Student, Weston College, Mass.

M. N. CHASE, M.A. (Illinois) Mathematician, Battelle Memorial Institute, Columbus, Ohio.

C. L. CHILDRESS, M.A. (Washington S. C.) Engineer, Boeing Airplane Company, Seattle, Wash.

T. Y. CHOW, Ph.D. (Cornell) Asst. Professor, Rensselaer Polytechnic Institute.

JOHN CHRISTOPHER, Ph.D. (Oregon) Instr., Knox College.

W. J. COLES, M.A. (Duke) Grad. Fellow, Duke University.

I. G. CONNELL, Student, University of Manitoba.

H. W. CONNORS, M.A. (South Dakota) Instr., University of South Dakota.

ALBERTO CORAZAO, D.Sc. (Peru) Professor, Escuela Militar de Chorrillos, Lima, Peru.

J. B. CORNELISON, B.A. (Berea) Part-time Instr., University of Kentucky.

R. T. DAMES, M.S. (Michigan) Research Assistant, Willow Run Research Center, Ypsilanti, Mich.

R L. DAVIS, M.S. (Chicago) Research Assistant, University of Michigan.

R. Y. DEAN, Ph.D. (C.I.T.) Mathematician, General Electric Company, Hanford Works, Richland, Wash.

T. F. DROEGE, B.S.E.E. (Kentucky) Grad. Assistant, University of Kentucky.

E. L. DUBOWSKY, B.S. in Educ. (Northwest Missouri S. C.) Grad. Assistant, Kansas State College.

R. W. EMMERT, M.A. (Ohio State) Instr., Miami University, Oxford, Ohio.

VELTA ERDMANIS, Student, University of Richmond.

MARY E. ESTILL, Ph.D. (Texas) Instr., Duke University.

O. J. FARRELL, Ph.D. (Harvard) Professor, Union College.

W. R. FERRANTE, B.S. (Rhode Island) Instr., Lafayette College.

R. J. FIORE, Student, University of Buffalo.

GLORIA C. FORD, M.A. (Pennsylvania) Instr., Virginia State College.

W. T. FORD, B.A. (Oklahoma City) Grad. Assistant, University of Oklahoma.

M. P. FRIEDMAN, B.S.C.E. (N.Y.U.) Instr., Brooklyn College.

T. A. GEORGEVITCH, Consultant, Department of Defense.

S. I. GOLDBERG, Ph.D. (Toronto) Instr., Lehigh University.

J. C. GOUSE, Student, University of Rochester.

J. J. GREEVER, III, Student, University of Richmond.

A. E. HALTEMAN, M.A. (Oregon) Asst. Professor, University of Idaho.

MRS. ALBERTA H. HENRY, Ph.D. (Michigan) Instr., Brooklyn College.

L. H. HERBACH, M.A. (Columbia) Instr., Brooklyn College.

MYRON HOCHHEISER, Student, City College of the City of New York.

MISS TOBALEE ISAACS, B.A. (Western Maryland) Part-time Instr., Duke University.

R. F. JACKSON, Ph.D. (Harvard) Asso. Professor, University of Delaware.

VERN JAMES, Ph.D. (Stanford) Professor, San Jose State College.

DOUGLAS JONES, Student, Oklahoma University.

J. G. JORDAN, M.A. (Michigan) Head of Department, Board of Education, Maumee, Ohio.

J. H. KAPLAN, Student, Temple University.

F. C. KOCH, Student, University of California.

LOUIS KRAMER, D.D.S. (Buffalo) Student, University of Buffalo.

R. E. KRUCKLIN, B.S. (M.I.T.) Senior Engineer, A. H. Johnson and Company, New York City.

ELENORE M. LAZANSKY, M.A. (California) Teacher, Oakland Public Schools, Oakland, Calif.

R. J. LEMELIN, M.A. (Bowling Green S. U.) Research Assistant, Willow Run Research Center, Ypsilanti, Mich.

J. L. LEWIS, Student, University of Notre Dame.

VIKTORS LINIS, M.Sc. (McGill) Instr., University of Saskatchewan.

C. H. LITTLE, JR., M.A. (North Carolina) Assistant Professor, North Carolina State College.

T. J. LITTLE, Student, University of Richmond.

L. S. LOCKINGEN, B.S. (Houston) Instr., University of Houston; Research Engineer, M. D. Anderson Hospital for Cancer Research.

A. R. LOVAGLIA, Ph.D. (California) Asst. Professor, San Jose State College.

W. M. LOWNEY, B.S. (Great Falls) Grad. Student, Montana State College.

J. S. MACNERNEY, Ph.D. (Texas) Asst. Professor, University of North Carolina.

H. V. MADISON, Student, Oklahoma City University.

W. G. MADOW, Ph.D. (Columbia) Professor, University of Illinois.

WILLIAM MARCACCIO, M.S. (Rhode Island) Asso. Professor, Xavier University, Cincinnati, Ohio.

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H. S. SHAPIRO, Ph.D. (M.I.T.) Member, Technical Staff, Bell Telephone Laboratories, Murray Hill, N. J.

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A. ZIRAKZADAH, M.S. (Oklahoma A. & M.) Instr., Oklahoma Agricultural and Mechanical College.

REPORT OF THE TREASURER FOR THE YEAR 1952

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1952. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1952

M & T Trust Co., Buffalo	$ 9,884.77	Current Fund	$ 9,884.77
Securities	61,594.98	Carus Fund	11,927.44
		Chace Fund	11,240.89
		Houck Fund	8,612.24
		Chauvenet Fund	653.40
		General Fund	29,161.01
	$71,479.75		$71,479.75

II. CURRENT FUND

Balance, January 1, 1952	$ 9,884.77	MONTHLY	
Dues	20,962.10	Publication	$17,472.60
Initiation fees	770.00	Reprints (net)	397.89
Subscriptions	4,475.34	Editor's Office	1,365.27
Sale of back numbers	1,239.66	Secretary-Treasurer's Office	
Advertisements	3,646.00	Clerical help	7,479.86
Sale of exchange periodicals	93.20	Postage and printing	1,877.50
Interest on General Fund	1,146.70	Office expenses	632.41
Income from Hardy Fund	120.00	Bank fees	65.21
Contributions	2.80	Auditing fee	25.00
Charges against Funds	858.95	Board of Governors	1,776.90
		Meetings	583.65
		Representatives	224.07
		Subventions	526.61
		Transfer to General Fund	1,401.74
		Balance, December 31, 1952	9,370.81

	III. Carus Fund	IV. Chace Fund	V. Houck Fund
Balance, January 1, 1952	$11,927.44	$11,240.89	$ 8,612.24
Sale of publications	2,695.59	641.40	5.00
Interest	470.09	442.39	339.67
Increase in value of securities	381.62	360.02	276.02
Less charges	675.51	163.81	18.40
Less Symposium expenses	—	—	600.00
Balance, December 31, 1952	$14,799.23	$12,520.89	$ 8,614.53

	VI. Chauvenet Fund	VII. Dunkel Fund	VIII. General Fund
Balance, January 1, 1952	$ 653.40	—	$29,161.01
Cash and securities from Dunkel Estate	—	$13,453.12	—
Interest	26.63	343.20	—
Increase in value of securities	21.60	429.63	931.25
Transfer from Current Fund	—	—	1,401.74
Less charges	1.23	—	—
Balance, December 31, 1952	$ 700.40	$14,225.95	$31,494.00

IX. Total Funds of the Association on December 31, 1952

Current Fund	$ 9,370.81	M & T Trust Co., Buffalo	$ 9,370.81
Carus Fund	14,799.23	Securities	82,355.00
Chace Fund	12,520.89		
Houck Fund	8,614.53		
Chauvenet Fund	700.40		
Dunkel Fund	14,225.95		
General Fund	31,494.00		
	$91,725.81		$91,725.81

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Delaware, Newark, Delaware, on November 29, 1952. Professor F. L. Manning, Chairman of the Section, presided at the morning and afternoon sessions.

Sixty persons were present, including the following thirty-nine members of the Association:

R. D. Anderson, R. J. Bickel, F. Marion Clarke, James Elmer Davis, F. L. Dennis, N. J. Fine, Mariano Garcia, E. D. Glenney, Samuel Goldberg, Emil Grosswald, V. H. Haag, Carl Hammer, Katharine E. Hazard, J. R. Holzinger, C. E. Kerr, R. A. C. Lane, Marguerite Lehr, E. V. Lewis, F. L. Manning, Edith A. McDougle, S. S. McNeary, C. A. Nelson, J. C. Oxtoby, C. F. Pinzka, G. E. Raynor, C. J. Rees, B. E. Rhoades, Judith A. Richman, J. D. Rutledge, C. W. Saalfrank, R. D. Schafer, Sister Mary Raphael, E. P. Starke, Rothwell Stephens, Alexander Tartler, J. I. Thigpen, H. G. Tucker, G. C. Webber, D. W. Western.

At the business meeting the following officers were elected for the coming year: Chairman, Professor C. A. Nelson, New Jersey College for Women; Secretary, Professor R. D. Schafer, University of Pennsylvania. The Program Committee for the next meeting will be: Professor Samuel Goldberg (Chairman), Lehigh University; Professor Alice T. Schafer, Drexel Institute of Technology; Professor D. W. Western, Franklin and Marshall College.

The program consisted of the following papers:

1. *Matrix inversion by partitioning*, by Professor Russell Remage, University of Delaware, introduced by the Secretary.

Computation of the inverse by Schur's method, of a non-singular $n \times n$ matrix A leads ordinarily to larger round off errors than, say, Gauss' method, while the usual iteration, $W^{(m+1)} = W^m(2 - AW^m)$, at the kth stage is very costly in terms of operations. It is found that iteration of only the last column requires fewer operations, by a factor of better than $2/k$, than a full iteration; yet the accuracy is at least as good. This makes Schur's method competitive with other methods, particularly for nearly singular matrices. The above technique is very useful for block partitioning as well, and for intermediate improvement of other methods.

2. *Probability models in engineering and biology*, by Professor Samuel Goldberg, Lehigh University.

Some techniques of the Mathematical Theory of Probability are applied to certain biological and engineering problems. The difference equation

$$f_{n+1} = 1 - (1 - \alpha)\frac{1 - f_n}{1 - \beta f_n^2} \qquad (n = 0, 1, 2, \cdots,\ 0 \leqq \alpha \leqq 1,\ 0 \leqq \beta \leqq 1),$$

is derived for f_n, the frequency of a gene in a specified population, as a function of the generation number n, and the parameters α and β which measure mutation and selection effects. Certain special cases of biological significance are considered and the sequence (f_n), $n = 1, 2, 3, \cdots$, is studied in the general case with respect to convergence to a limiting equilibrium value and the dependence of this equilibrium on the initial frequency f_0. A problem in strength of materials which leads to a simple Markov chain is also discussed.

3. *The Ramanujan identities*, by Professor N. J. Fine, University of Pennsylvania.

Ramanujan's identities for the partition-generating functions $\sum p(5n+4)x^n$ and $\sum p(7n+5)x^n$ exhibit explicitly the divisibility of the coefficients by 5 and 7 respectively. An alternate proof of the first, by Ramanujan, has recently come to light. The author extends these methods to obtain an identity for the second function, differing from Ramanujan's, but serving the same purpose. Watson's use of modular equations to prove the congruences for 5^α and 7^α is mentioned, and a new method is sketched for obtaining these equations systematically. Although the derivation of the modular equation for the modulus 11 is not complete, enough has been done to obtain an identity of Ramanujan type, hence an alternate proof of the congruence $p(11n+6) \equiv 0 \pmod{11}$.

4. A. *An in-service program in statistics*; B. *Some research opportunities in basic mathematics*, by Professor E. V. Lewis, University of Delaware.

A. The Unit Programs in Statistics offered to industry by the University of Delaware provide training in the use of groups of statistical methods specially selected for each contracting firm. The training consists of three parts: (1) a detailed discussion of the methods presented for the main group of workers and their immediate supervision, (2) availability as consultants for a stated

period during the program, and (3) a discussion with management and supervision of the use of these methods in their firm. The benefits of the program to both industry and the University were discussed and suggestions were offered for programs in other fields.

B. The essential features of several types of research effort were illustrated by analogies. Examples of problems which have remained unsolved while mathematics has grown around and beyond them were discussed. In the case of each example, useful approximation methods exist, but the absence of an exact solution is a barrier to progress in a related science. These specific problems were then considered as examples of ways in which other opportunities for research may be uncovered.

R. D. SCHAFER, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Howard University, Washington, D. C., on December 6, 1952. Professor Marian M. Torrey, Chairman of the Section, presided at the morning and afternoon sessions.

There were one hundred fifteen persons in attendance, including the following eighty-two members of the Association:

J. C. Abbott, M. I. Aissen, D. F. Atkins, T. J. Benac, T. A. Botts, Evelyn Boyd, B. H. Buikstra, W. E. Byrne, H. H. Campaigne, L. H. Chambers, H. J. Cheston, Jr., F. Marion Clarke, G. R. Clements, L. S. Dederick, J. W. Drew, J. A. Duerksen, E. L. Eagle, W. L. Fields, G. C. Francis, C. H. Frick, S. I. Gass, Michael Goldberg, R. A. Good, R. D. Gordon, E. S. Grable, E. C. Gras, Donald Greenspan, D. W. Hall, M. Gweneth Humphreys, S. B. Jackson, F. E. Johnston, Sidney Kaplan, L. M. Kells, F. B. Key, Blair Kinsman, H. L. Kinsolving, Mary A. Lee, D. C. Lewis, Jr., D. B. Lloyd, D. B. Lowdenslager, Elizabeth C. Lukacs, C. H. McCall, Jr., Carol V. McCamman, E. J. McShane, J. F. Manogue, Ella Marth, Florence M. Mears, Joseph Milkman, George Millman, R. W. Moller, Dewey Moore, T. W. Moore, W. K. Morrill, W. R. Murray, E. J. Musch, W. H. Norris, Jr., M. W. Oliphant, Hyman Orlin, J. W. Ponds, J. W. Popow, O. J. Ramler, R. W. Rector, J. D. Riley, T. J. Rivlin, J. B. Scarborough, W. F. Shenton, C. H. Sisam, H. C. Stotz, E. H. Swafford, Mildred E. Taylor, Feodor Theilheimer, O. M. Thomas, P. D. Thomas, J. A. Tierney, Marian M. Torrey, John Tyler, C. Y. Wang, M. E. White, P. M. Whitman, J. W. Wrench, Jr., D. M. Young, G. C. Zader.

During the business meeting it was announced that the Section will sponsor contests for high school students in order to stimulate further interest in mathematics.

The following papers were presented:

1. *Multilinear forms, n-ics and polynomials on a Hilbert space*, by Professor Joseph Milkman, United States Naval Academy.

Multilinear forms and n-ics on a Hilbert space were defined. Methods of finding the inner product of n-ics, polynomials, and power series were shown. The space C of all polynomials and power series of finite norms is a Hilbert space. The functionals discussed in this paper have been applied to the quantum theory of fields by K. O. Friedrichs.

2. *A new configuration associated with the osculating quadrics at a point of an analytic surface*, by Mr. P. D. Thomas, Ballistic Research Laboratories.

Certain lines in the tangent plane at a given point of an analytic surface were found as the result of certain mutual tangent plane relations between a quadric of Darboux and Bompiani's asymptotic osculating quadrics. The polar reciprocals of these lines in the tangent plane trace certain loci on a quadric of Darboux. The investigation of these loci leads to a new geometric configuration associated with a point of an analytic surface. The treatment was projective, using homogeneous coordinates.

3. *Elementary techniques in maxima and minima*, by Professor J. A. Tierney, United States Naval Academy.

The application of the first derivative in the solution of maxima minima problems is an important topic in a first course in calculus. It is interesting to note that many of the standard textbook problems of this type can be solved without the use of the calculus and require only a knowledge of the mathematics customarily taught in the secondary schools. The elementary techniques which are effective vary considerably, some being trivial in nature and others requiring a certain amount of ingenuity, whereas the calculus approach is usually straightforward. Several examples illustrating different techniques were presented.

4. *The review course in calculus at the United States Department of Agriculture Graduate School*, by Mr. Sidney Kaplan, Office, Comptroller of the Army and United States Department of Agriculture Graduate School.

The work is condensed to forty-five hours of work. Six derivatives are developed: the derivative of a constant, the derivative of a function of a function, the derivative of a sum of functions, the derivative of an inverse function, the derivative of a sine function, and the derivative of a logarithmic function. Four topics are stressed as being more essential from the applied point of view: curve tracing, differential—approximation and estimate of error, Newton-Raphson Method, and the trapezoidal and Simpson's mechanical quadrature formulae.

5. *The Stieltjes integral*, by Professor E. J. McShane, University of Virginia.

This was the invited lecture. It was pointed out that the Stieltjes integral is an appropriate device for use in many applications; for example, in investigating moments and moments of inertia of mass distributions, where the distribution may include point masses and also a continuous part; in studying the statistics associated with probability distributions; and in expressing linear functions, such as the differential effects encountered in exterior ballistics. The particular version of the Stieltjes integral presented was that due to McShane and Botts (*Duke Mathematical Journal*, vol. 19, 1952, p. 293), which has some advantages not possessed by the integral as usually defined.

C. H. FRICK, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, August 31–September 1, 1953.

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May 2, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS, Washburn Municipal University of Topeka, April 11, 1953.

KENTUCKY, University of Louisville, May 9, 1953.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA United States Naval Proving Ground, Dahlgren, Virginia, May 2, 1953.

METROPOLITAN NEW YORK

MICHIGAN, Wayne University, Detroit, April 18, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, April 24, 1953.

NEBRASKA, University of Nebraska, Lincoln, May 2, 1953.

NORTHERN CALIFORNIA

OHIO, Columbus, April 18, 1953

OKLAHOMA, Oklahoma City, October, 1953.

PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.

PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.

ROCKY MOUNTAIN, University of Colorado, Boulder, April 17–18, 1953.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Fort Worth, April 24–25, 1953.

UPPER NEW YORK STATE, United States Military Academy, West Point, May 9, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May 2, 1953.

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No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

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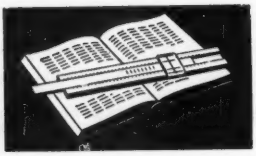

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